EFFECTS OF COUPLE STRESSES ON THE STABILITY OF PLANE POISEUILLE FLOW

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CERTIFICATE

This is to certify that this work has been carried out under my supervision and has not been submitted for a degree elsewhere.

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1. INTRODUCTION

1.1 The theory of laminar instability is of great practical interest because it provides qualitative criteria for the important problem of laminar-turbulent transition. It has been observed experimentally that, laminar flow occurs at low Reynolds numbers and that in this range viscosity damps out any deviations from laminar flow. On the other hand, turbulent motion occurs at high Reynolds numbers. Laminar flow can be realised, if at all, only by excluding all possible disturbances. One is led to suspect therefore that there might be more than one solution to the equations of fluid motion. The stability theory at least explains the non occurrence of laminar flow under certain combinations of flow parameters. At present, the predictions of stability theory are mainly limited to the description of the initial breakdown of the laminar flow due to infinitesimal disturbances. It is still a long way from explaining the regime of fully developed turbulent flow. Stability theory also predicts the qualitative effectiveness of the various flow parameters in promoting or suppressing stability. The fundamental aspects of the theory of laminar stability have been summarised in a monograph by Lin[1].

1.2 The classical theories of mechanics do not take couple stresses into account. This essentially means that the mechanical interaction between portions of a body across a surface in it can be represented by distributed forces only. In recent years much attention has been paid to polar theories which take into account couple stresses. Stokes[2] has considered one such polar theory for fluids and has studied the effects of couple stresses by solving some representative boundary value problems for fully developed viscometric flows. The results show that an interesting phenomenon involving a size dependent effect, which is not present in the classical nonpolar theory, comes in.

In this thesis, an attempt has been made to study the effects of couple stresses on the stability of parallel flows. Numerical results have been obtained for plane Poiseuille flow.

1.3 The mathematical problem of hydrodynamic stability can be formulated by using the given steady state solution of the equations of motion and superimposing on it a disturbance of suitable kind. This results in a set of non-linear disturbance equations which govern the behaviour of the disturbance. If the disturbance ultimately decays to zero as time tends to infinity, the flow is said to be

stable. But if a disturbance results which is permanently different from zero, the flow is said to be unstable. It does not follow that instability leads to turbulent motion, because another probably more complex form of the laminar motion may be the result. We assume that for small disturbances the disturbance equations may be linearised. In general, the resulting equations are homogeneous with homogeneous boundary conditions. This leads to a characteristic value problem with a parameter σ. If all the characteristic values of σ have negative real parts, the motion is said to be stable with respect to infinitesimal disturbances. If some of the characteristic values of σ have positive real parts, the flow is said to be unstable. Neutral stability is marked by zero values of the real part of σ. The possible eigenvalues of σ depend, of course, on quantities such as flow speed, kinematic viscosity, thermal diffusivity and the wave length of the disturbance.

A brief account of the development of polar theories is given in the next section.

1.4 The classical theory of continua models properties of media in which a central force acts between particles. The inadequacy of this theory, for some problems, led to the consideration of theories of continuous media which take

into account rotational interaction amongst particles. This gave rise to the concept of couple stresses. It is well known that the action, at a point, of even a simple system of forces cannot be reduced to a single resultant force, but is in general equivalent to a force and a couple. Consequently, to describe the mechanical interaction between a system of particles, it is necessary in general to consider forces and moments.

The literature on couple stresses begins with the memoir of Cosserat and Cosserat[3]. The Cosserats gave a systematic development of the mechanics of continuous media each of whose points have the six degrees of freedom of a rigid body. In classical elasticity, a material point has only three degrees of freedom.

A striking feature of Cosserats' theory was the appearance of couple stresses in the equations of motion and the resulting asymmetry of the stress tensor. A modern derivation of the Cosserat equations was given by Truesdell and Toupin[4]. Mindlin and Tiersten[5] gave an extensive analysis of the infinitesimal motions of a Cosserat medium with constrained rotations. The mechanics of elastic media with micro-structures has been discussed by Mindlin[6]. Eringen[7] and his coworkers developed a theory for

microscopic materials which exhibit certain microscopic effects arising from the local structure and micromotions of the media. Eringen[8] also simplified this general theory to the micro polar theory and solved some boundary value problems. Upadhyaya[9,10] studied the effects of couple stresses in linear viscoelasticity.

1.5 Many theories of polar fluids have been developed. A polar fluid is a fluid in which couple stresses are present. In the general polar fluid theories, the angular velocity of a fluid particle is allowed to differ from the average angular velocity of its neighbourhood (which is called the vorticity) by introducing an additional independent angular velocity (or internal spin) vector field. By constraining the rotation of a fluid particle to equal the local rotation of the medium, the couple stress theory results as a special case of the general polar fluid theory.

The theory of polar fluids is analogous to the theory of polar elasticity in the same way as the classical theory of elasticity is analogous to the classical theory of fluids. Grad[11](1952) first introduced the constitutive equations for a polar fluid. These constitutive equations relate the skew symmetric part of the usual stress tensor to the relative angular velocity of a fluid particle (with respect to its

neighbourhood), and the couple stress tensor to the gradient of the total or particle angular velocity. The Newtonian law of viscosity is preserved by assuming that the usual relationship between the symmetric part of the stress tensor and the rate of deformation tensor holds. Aero, Bulygin and Kuvshinkii[12] introduced the velocity and total angular velocity fields and postulated a dissipation function that led to Grad's constitutive equations. Cowin[13] (1962) used a Cosserat continuum in which a rigid triad of vectors is associated with each particle. The total or particle angular velocity is then represented by the angular velocity of the rigid triad and leads to the same constitutive equations. Eringen[14] (1964) developed the theory for micro-fluids. By specialising the micro-fluid theory to the theory of micro-polar fluids, Eringen[15] obtained constitutive equations similar to those of Grad. Dahler[16] developed the same constitutive equations from a statistical mechanics approach.

Stokes obtained a linear theory for fluids with couple stresses parallel to the couple stress theory in linear elasticity proposed by Mindlin and Tiersten. The symmetric part of stress tensor and the rate of deformation tensor follow the same relationship as in the non-polar

case. The couple stress tensor is assumed to be proportional to the curvature twist tensor which is defined to be vorticity gradient. This theory is a special case of the general polar theory, obtained by constraining the rotation of a point in a Cosserat continuum to be equal to the local rotation of the medium. The antisymmetric part of the stress tensor and the trace of the couple stress tensor are left undetermined by these constitutive equations but can be determined from the boundary conditions. Condiff and Dahler[17] and Pennigton[18] have considered plane Couette flow, plane Poiseuille flow, Couette flow between two concentric cylinders and Poiseuille flow in a pipe. Cowin[13] considered plane Poiseuille flow, Eringen and many others have considered the flow of micro-polar fluids in different flow situations. Stokes has solved Couette and Poiseuille flows between parallel plates, flow between concentric cylinders with and without a toroidal pressure gradient and Poiseuille flow in a pipe. Stokes[19, 20] also considered the effects of couple stresses in fluids on hydromagnetic flow in a channel and on heat transfer. Dep[21] has studied the equations for fluid boundary layers with couple stresses. Dep's analysis is based on the equations given by Aero, Bulygin and Kuvshinkii. Usmani[22] considered the time dependent effects in fluids

with couple stresses, using the theory proposed by Stokes.

In this thesis an attempt has been made to study the effects of couple stresses on the stability of parallel flows. It has been assumed that the fluid is incompressible and that the body forces and body moments are absent. The equations of motion are fourth order partial differential equations. The method of normal modes has been used to study the variation of disturbances with time.

The disturbance equation comes out to be a sixth order ordinary linear differential equation subject to homogeneous boundary conditions. This leads to an eigenvalue problem. The stability of parallel flows has been analysed for three-dimensional and two-dimensional disturbances. It has been observed that the minimum critical Reynolds number is obtained in the case of two-dimensional disturbances. Keeping this in view, stability of plane Poiseuille flow has been studied numerically. The initial value technique suggested by Nachtshiem[36] has been used to solve the characteristic value problem. The integration is done by a step-by-step integration method. To have better control on truncation error, the fifth order Milne predictor-corrector integration technique is used. Double precision arithmetic (16 significant digits) has been used to control round-off

errors. The effects of couple stresses on the Reynolds number, on the disturbance wave number and on the stability factor have been studied. Neutral stability curves for different values of couple stress parameter have been obtained. The minimum critical Reynolds number in each case has been obtained.

2. HYDRODYNAMIC STABILITY WITH COUPLE STRESSES

Following Stokes[2], the theory of couple stresses in fluids is briefly discussed in this section.

2.1 KINEMATICS OF FLOW :

Let u_i be the components of the velocity field. Then, the rate of deformation tensor D_{ij} and the vorticity tensor W_{ij} are given by,

$$D_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j}) \tag{2.1.1}$$

$$W_{ij} = \frac{1}{2}(u_{j,i} - u_{i,j}) \tag{2.1.2}$$

respectively.

The vorticity vector is defined as half the curl of the velocity vector, so that

$$\omega_i = \frac{1}{2} e_{irs} u_{s,r} \tag{2.1.3}$$

The vorticity vector and the vorticity tensor are related by the equations

$$W_{ij} = e_{ijr} \omega_r \tag{2.1.4}$$

and $$\omega_i = \frac{1}{2} e_{irs} W_{rs} \tag{2.1.5}$$

The curvature-twist rate tensor K_{ij} is defined to be the gradient of the vorticity field; thus,

$$K_{ij} = \omega_{j,i} \tag{2.1.6}$$

The diagonal components of K_{ij} are a measure of the rate of twist of the material per unit length about the three axes. On the other hand, the off diagonal elements of K_{ij} are a measure of the curvatures induced per unit time in the various planes.

Equations (2.1.3) and (2.1.6) give,

$$K_{ii} = 0 \tag{2.1.7}$$

That is, the average twist rate per unit length of three mutually perpendicular line elements is zero.

2.2 CONSTITUTIVE EQUATIONS:

In the non-polar case there is a close similarity between the elastic and fluid constitutive equations. Following Mindlin and Tiersten's linear constitutive equations for polar elasticity Stokes[2] proposed the following constitutive equations for fluids,

$$T^{S}_{ij} = - p\,\delta_{ij} + \lambda\, D_{rr}\,\delta_{ij} + 2\mu D_{ij} \tag{2.2.1}$$

$$M^{D}_{ij} = 4\,\eta\, K_{ij} + 4\,\eta'\, K_{ji} \tag{2.2.2}$$

where the stress tensor T_{ij} and the couple stress tensor M_{ij} are given by the equations,

$$\begin{aligned} t_i &= n_j\, T_{ji} \\ m_i &= n_j\, M_{ji} \end{aligned} \tag{2.2.3}$$

where n_j is the unit normal to the surface on which t_i and m_i act.

It is clear from equations (2.2.1) and (2.2.2) that, for couple stresses to exist, the vorticity gradient field must be non zero. The dimensions of the material constants λ and μ are those of viscosity (namely M/LT), whereas the dimensions of η and η' are those of momentum (namely ML/T). The ratio η/μ has dimensions of length square. This material constant is denoted by ℓ^2 where

$$\ell = (\eta/\mu)^{1/2} \tag{2.2.4}$$

For incompressible fluids $u_{r,r} = 0$, and in the absence of body forces and body moments, Cauchy's law of motion governing the balance of linear momentum is given by,

$$\rho \dot{u}_i = -p_{,i} + \mu u_{i,rr} - \eta u_{i,rrss} \tag{2.2.5}$$

2.3 BOUNDARY CONDITIONS:

In dealing with this polar theory six boundary conditions are necessary. It seems quite reasonable to assume the no-slip condition at the boundary as in the non-polar case. The formulation of additional boundary conditions is restricted to certain idealised cases only, because the mechanism of interaction at the boundary is not understood clearly. In the general theory of polar fluids the most commonly used boundary conditions are for limiting case wherein a solid surface interacts so strongly with the neighbouring fluid, that a particle of fluid does not turn over relative to the surface, and therefore, its

angular velocity is equal to the angular velocity of the surface. This makes use of the property of adherence of a fluid to a solid surface. In the constrained polar theory, this condition becomes trivial, as the particle and the local angular velocities are assumed to be same throughout the flow field.

To prescribe additional independent boundary conditions in this case, use is made of the other idealised situation wherein the solid surface allows rotational slip of a fluid particle in such a way that the couple stress vector is zero at the surface. This leads to boundary condition :

$$m_i = n_j M_{ji} \big|_{surface} = 0$$

where n_j is the unit normal to the surface.

Following Stokes[2] the condition of no couple stresses at the solid boundaries is used here.

2.4 GOVERNING EQUATIONS IN NON-DIMENSIONAL FORM :

For incompressible fluids, in the absence of body forces and body moments, the velocity components u_i^* and the pressure p^* satisfy the equation of continuity,

$$u^*_{i,i} = 0 \tag{2.4.1}$$

and the equations of motion, given by equations (2.2.5), namely,

$$\begin{aligned} \rho^* \dot{u}_i^* &= - p^*_{,i} + \mu^* u^*_{i,rr} - \eta^* u^*_{i,rrss} \\ p^* &= p^* (x_i^*) \end{aligned} \tag{2.4.2}$$

The asterisk indicates that the quantity concerned is dimensional. The superposed dot indicates the derivatives with respect to time following a particle, and ρ is the mass density of the material.

It is convenient to work entirely in terms of non-dimensional quantities. For this purpose, following characteristic quantities are introduced :

L^* = characteristic length (length scale)

V^* = characteristic velocity (velocity scale)

L^*/V^* = characteristic time (time scale)

$\rho^* V^{*2}$ = characteristic pressure (pressure scale)

The non-dimensional quantities may be obtained from dimensional quantities by making use of the simple relation :

$$\text{Non-dimensional quantity} = \frac{\text{Dimensional quantity}}{\text{scale}}$$

The non-dimensional variables are defined by,

$$t = \tau^* / L^* / V^*$$

$$x_i = x_i^* / L^*$$

$$u_i = u_i^* / V^*$$

$$p = p^* / \rho^* V^{*2}$$

The equations (2.3.1) and (2.3.2) may be written in non-dimensional form as,

$$u_{i,i} = 0 \qquad (2.4.3)$$

and

$$\frac{\partial u_i}{\partial t} + u_r\, u_{i,r} = -p_{,i} + \frac{1}{R}\, u_{i,rr} - \frac{1}{RK}\, u_{i,rrss} \qquad (2.4.4)$$

The non-dimensional parameters R and K are given by,

$$R = V^* L^* \rho^* / \mu^* \qquad (2.4.5)$$
$$K = L^{*2} / \ell^2$$

Here R resembles the Reynolds number in non-polar theory for fluids.

At solid boundaries with prescribed velocities S_i and couple stress tensor C_{ij} the conditions

$$u_i = S_i$$

and $\qquad\qquad$ (2.4.6)

$$m_i = n_j\, C_{ji}$$

must be satisfied.

If the boundary velocities S_i and couple stress C_{ij} are independent of time, we may have a steady motion given by,

$$u_i = \bar{u}_i\,(x_k)$$
$$p = \bar{p}\,(x_k) \qquad (2.4.7)$$
$$n_j \bar{M}_{ji} = n_j\, \bar{M}_{ji}(x_k)$$

Satisfying equations (2.3.3) and (2.3.4) in the form,

$$\bar{u}_i\, \bar{u}_{i,j} = -\,\bar{p}_{,i} + \frac{1}{R}\, \bar{u}_{i,rr} - \frac{1}{RK}\, \bar{u}_{i,rrss} \qquad (2.4.8)$$

$$\bar{u}_{k,k} = 0 \qquad (2.4.9)$$

together with the boundary conditions.

$$\bar{u}_i = S_i$$
$$n_j \bar{M}_{ji} = n_j \, C_{ji} \tag{2.4.10}$$

5. MATHEMATICAL FORMULATION OF THE STABILITY PROBLEM FOR AN INCOMPRESSIBLE FLUID :

To study the effects of small disturbances on the steady motion, we seek solutions of equations (2.4.3) and (2.4.4) in the form

$$u_i = \bar{u}_i + \epsilon \, u_i^{(1)} + \epsilon^2 \, u_i^{(2)} + \ldots\ldots\ldots\ldots\ldots$$
$$p = \bar{p} + \epsilon \, p^{(1)} + \epsilon^2 \, p^{(2)} + \ldots\ldots\ldots\ldots\ldots\ldots \tag{2.5.1}$$

where ϵ is a constant parameter and $u_i^{(1)}$, $u_i^{(2)}$, ... $p^{(1)}$, $p^{(2)}$, ... are functions of position and time. We demand that equations (2.5.1) satisfy equations (2.4.3), (2.4.4) and (2.4.6) for all values of ϵ in the range $0 < \epsilon < \epsilon_1$. Formal substitution from equation (2.5.1) into equations (2.4.3), (2.4.4) and (2.4.6) gives a sequence of sets of four equations, each set corresponding to a definite power of ϵ. The set corresponding to ϵ^0 is given by equations (2.4.8) and (2.4.9) along with the boundary conditions given by equation (2.4.10). The set corresponding to ϵ^1 is

$$\frac{\partial u_i^{(1)}}{\partial t} + \bar{u}_j \, u_{i,j}^{(1)} + u_j^{(1)} \, \bar{u}_{i,j} = - p_{,i}^{(1)} + \frac{1}{R} \, u_{i,rr}^{(1)} - \frac{1}{RK} u_{i,rrss}^{(1)} \tag{2.5.2}$$

and the continuity equation is

$$u_{i,i}^{(1)} = 0 \tag{2.5.3}$$

The boundary conditions for this set are

$$\left.\begin{aligned} u_i^{(1)} &= 0 \\ n_j\, M_{ji} &= 0 \end{aligned}\right\} \quad \text{on solid boundaries.}$$

A complete treatment would require consideration not only of equations (2.5.2) but also of the equations corresponding to higher powers of ϵ, together with the consideration of the convergence of equations (2.5.1) and a justification of the term-by-term differentiation. The problem of hydrodynamic stability reduces to the problem of solving a system of non-linear (quasi-linear) partial differential equations.

In the usual approaches to the problem, the mathematical formulation is cast in a different way. We assume that, for small disturbances the equations may be linearised; that is, we neglect quadratic or higher order terms in the disturbances and their derivatives. Thus, the linearised theory corresponds to equation (2.5.2). The linearised theory of stability of laminar flows decomposes the motion into a mean flow (whose stability constitutes the subject of the investigation) and into a disturbance superimposed on it.

Let the mean flow be given by $\overline{u}_i$ and pressure by $\overline{p}$. The corresponding quantities for the non-steady disturbance will be denoted by u_i' and p' respectively. The resulting motion is given by,

$$u_i = \overline{u}_i + u_i' \tag{2.5.5}$$

$$p = \overline{p} + p' \tag{2.5.6}$$

The couple stress tensor may be written as,

$$M_{ij} = \overline{M}_{ij} + M_{ij}' \tag{2.5.7}$$

It is assumed that the disturbance quantities are small compared with the corresponding quantities of the mean flow.

The investigation of the stability of such a disturbed flow can be carried out with the aid of either of two different methods. The first method (Energy method) consists merely of calculating the variation of the energy of the disturbance with time. Conclusions are then based on whether the energy decreases (the flow is stable) or increases (the flow is then unstable) with time. The theory admits an arbitrary form of the superimposed motion and demands only that it should be compatible with the equation of continuity. In general, the energy method has not proved successful. The second method accepts only flows in which the steady state and total velocity fields satisfy

the equations of motion. This is the method of small disturbances and has proved successful.

It is assumed that the mean flow given by equation (2.4.7) is a solution of equation (2.4.4) and that the resultant motion, governed by equations (2.5.5), (2.5.6) and (2.5.7), also satisfy this equation. Substituting for the resultant motion in the equation (2.4.4) and linearising, we get

$$\frac{\partial u_i'}{\partial t} + \bar{u}_r u'_{i,r} + u_i' \bar{u}_{i,r} = - p'_{,i} + \frac{1}{R} u'_{i,rr} - \frac{1}{RK} u'_{i,rrss} \qquad (2.5.8)$$

The continuity equation becomes

$$u'_{i,i} = 0 \qquad (2.5.9)$$

The boundary conditions reduce to,

$$\left.\begin{aligned} u_i' &= 0 \\ n_j M'_{ji} &= 0 \end{aligned}\right\} \qquad (2.5.10)$$

A study of the linear system of equations (2.5.8) shows that the time appears only through the derivatives with respect to time t and hence solutions containing an exponential time factor $e^{\sigma t}$ may be expected. Since the original boundary conditions are already satisfied by the steady motion, the disturbance solutions must vanish at the boundaries. This

will be possible only for special values of σ. Thus we have a characteristic value problem with σ as a parameter. u_i', p' and M_{ij}' may be referred to as the modes of oscillations. Usually, by slightly changing some of the flow parameters, such as the Reynolds number, one finds slightly different eigen solutions for u_i', p_i' and M_{ij}' by allowing σ to be complex. If the complex eigenvalue σ has a positive real part, the solutions u_i', p' and M_{ij}' are amplified because they tend to infinity as t increases and instability is indicated. If σ has a negative real part, the solution is 'damped' and the steady flow is stable. The transition from stability to instability, in such cases, is marked by the vanishing of the real part of σ. It may be noted, however, that a degenerate case of the transition from stability to instability occurs when the real and imaginary parts of σ vanish simultaneously. The neutral mode then corresponds to a 'secondary flow' rather than to a 'steady oscillation'. In certain problems, it is assumed that the imaginary part of σ always vanishes if the motion becomes unstable. This can be shown explicitly for some problems. The secondary flow simply diverges with time after the stability boundary is crossed. In this case, one locates the stability boundary by finding the conditions for the

existence of steady secondary flow.

We shall formulate the stability problem for parallel flows in next section.

3. STABILITY OF PARALLEL FLOWS

3.1 FORMULATION OF STABILITY PROBLEM FOR PARALLEL FLOWS

Let the fluid occupy the region between two infinite parallel plates, which are either stationary or moving parallel to each other with uniform speeds.

Take a cartesian coordinate system with the x axis parallel to the direction of motion and the y and z directions perpendicular to it. The plates are situated at $y = y_1$ and $y = y_2$.

The mean velocity field is chosen as,

$$\bar{U} = \bar{u}(y),\ 0,\ 0 \tag{3.1.1}$$

The disturbance field is defined as,

$$U' = (u', v', w') \tag{3.1.2}$$

$$p = \bar{p} + p' \tag{3.1.3}$$

where $\bar{p} = \bar{p}(x)$

The disturbance quantities are functions of x, y, z and time t. The governing equations (2.5.8) and (2.5.9), for the flow situation governed by equations (3.1.1) and (3.1.2), reduce to

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + v'\frac{\partial \bar{u}}{\partial y} = -\frac{\partial p'}{\partial x} + \frac{1}{R}\nabla^2 u' - \frac{1}{RK}\nabla^4 u'$$

$$\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} + \frac{1}{R}\nabla^2 v' - \frac{1}{RK}\nabla^4 v'$$

$$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z} + \frac{1}{R}\nabla^2 w' - \frac{1}{RK}\nabla^4 w' \qquad (3.1.4)$$

where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

and $\nabla^4 \equiv \nabla^2 \nabla^2$

The boundary conditions for the system of equations (3.1.4) become

$$\left.\begin{aligned} u' &= 0 \\ v' &= 0 \\ w' &= 0 \\ n_j M'_{ji} &= 0 \end{aligned}\right\} \quad \text{at} \quad y = y_1 \text{ and } y_2 \qquad (3.1.5)$$

The precise discussion of this problem presents certain difficulties. Physically speaking, no experiments are performed with an apparatus of infinite dimensions. Mathematically, it is possible to circumvent the difficulty by limiting the discussion to disturbances which are spatially periodic in the directions in which the fluid extends to infinity. This is suggested by the nature of equations (3.1.4). Since all the three variables x, z, t are cyclic and since the coefficients of equations (3.1.4) depend on y only, the system admits solutions which are exponential

functions in x and z, as well as in t. If the solution is to be bounded for x and z both tending to $+\infty$ and $-\infty$, the corresponding exponents must be pure imaginary. Thus, each component of the disturbance is the real part of an expression of the form

$$q' = \hat{q}(y) \exp \left\{ i(\alpha x + \beta z) - i\alpha ct \right\}$$
$$-i\alpha c = \sigma^* L^*/V^* \qquad (3.1.6)$$

where α and β are real. This type of representation is generally referred to as the 'method of normal modes'. The three dimensional disturbances given by equations (3.1.6), having all the three components may be interpreted as oblique plane waves travelling in a direction making an angle of $\tan^{-1}(\beta/\alpha)$ with the main flow direction.

Since the directions are arbitrary, it is convenient to choose them in such a way that, the x axis coincides with the direction of the wave front.

The complex non-dimensional characteristic value σ has the value $\sigma = -i\alpha c$, in equation (3.1.6). The real wave numbers α and β are given by,

$$\alpha = 2\pi/\lambda_x$$

$$\beta = 2\pi/\lambda_z$$

where λ_x and λ_z are non-dimensional wave lengths in the x and z directions respectively.

3.2 STABILITY CRITERIA

In the expression given by equation (3.1.6) the complex quantity c may be written as

$$c = c_r + i\, c_i$$

where the real part c_r represents the wave velocity of the disturbance and the imaginary part c_i, the amplification or the damping of oscillations with time. The eigenvalue σ may be written as

$$\sigma = \sigma_r + i\sigma_i$$

$$= \alpha c_i - i\alpha c_r$$

As discussed in section 2.5, the stability criteria can be based on the real part of σ as,

(1) $c_i > 0$ the flow is unstable,

(2) $c_i = 0$ neutral stability,

and (3) $c_i < 0$ the flow is stable.

3.3 TWO DIMENSIONAL AND THREE DIMENSIONAL DISTURBANCES

From equations (3.1.6), the velocity and pressure terms may be written as

$$\{u', v', w', p'\} = \text{Real part } \{\hat{u}(y), \hat{v}(y), \hat{w}(y), \hat{p}(y)\}\{\exp[i(\alpha x + \beta z) - i\alpha ct]\} \tag{3.3.1}$$

The equations (3.1.4) can be simplified by substituting the values from equations (3.3.1). The simplified equations come out to be,

$$\left[\frac{\bar{\bar{L}}}{R} - i\alpha\,(\bar{u} - c)\right]\hat{u} = (D\bar{u})\,\hat{v} + i\alpha\,\hat{p}$$

$$\left[\frac{\bar{\bar{L}}}{R} - i\alpha\,(\bar{u} - c)\right]\hat{v} = D\hat{p} \qquad (3.3.2)$$

$$\left[\frac{\bar{\bar{L}}}{R} - i\alpha\,(\bar{u} - c)\right]\hat{w} = i\beta\hat{p}$$

$$i(\alpha\hat{u} + \beta\hat{w}) + D\hat{v} = 0$$

where the operators are defined by,

$$D \equiv d/dy$$

$$\bar{L} \equiv D^2 - (\alpha^2 + \beta^2) \qquad (3.3.3)$$

and $\bar{\bar{L}} = \bar{L}\,(1 - \bar{L}/K)$

The above equations hold for three dimensional disturbances. Following Squire (1933), we shall now show that the problem of the three dimensional disturbances is actually equivalent to a two dimensional problem at a lower Reynolds number. In fact, if we introduce the following transformations,

$$\tilde{\alpha}\tilde{u} = \alpha\hat{u} + \beta\hat{w}$$

$$\tilde{v} = \hat{v}$$

$$\tilde{p}\,\tilde{R} = \hat{p}R$$

$$\tilde{\alpha}\tilde{R} = \alpha R$$
$$\hat{c} = c \qquad (3.3.4)$$
$$\tilde{\alpha}^2 = \alpha^2 + \beta^2$$

The equations (3.3.2) may be written as

$$\left[\hat{\tilde{L}} - i\tilde{\alpha}\tilde{R}\,(\bar{u} - c)\right]\hat{\tilde{u}} = \tilde{R}\,(D\bar{u})\,\hat{\tilde{v}} + i\,\tilde{\alpha}\tilde{R}\tilde{P}$$
$$\left[\hat{\tilde{L}} - i\tilde{\alpha}\tilde{R}\,(\bar{u} - c)\right]\hat{\tilde{v}} = \tilde{R}\,D\,\hat{\tilde{p}} \qquad (3.3.5)$$
$$i\tilde{\alpha}\hat{\tilde{u}} + D\,\hat{\tilde{v}} = 0$$

where $\quad \hat{\tilde{L}} \equiv (D^2 - \tilde{\alpha}^2)\;(1 - (D^2 - \tilde{\alpha}^2)/K)$

This system of equations has the same structure as equations (3.3.2) with $\hat{w} = 0, \beta = 0$. That is, equations (3.3.5) correspond to a two dimensional disturbance. The boundary conditions obviously reduce to,

$$\hat{\tilde{u}} = \hat{\tilde{v}} = 0$$
$$n_j\,\hat{M}_{ji} = 0 \qquad \text{at } y = y_1 \text{ and } y_2 \qquad (3.3.6)$$

Thus, each three dimensional problem is equivalent to a two dimensional problem. Hence it is sufficient to solve two dimensional problems only. Three dimensional solutions may then be obtained by using the transformations indicated by equations (3.3.4). In fact equations (3.3.4) show that the equivalent two dimensional problem is associated with a lower Reynolds number, since $\tilde{\alpha} > \alpha$. Thus, the minimum critical Reynolds number is given directly by a two dimensional analysis.

Another method of arriving at the above result is physically more suggestive. The three dimensional disturbance is essentially an oblique wave. Now, if the x axis is chosen to be such that $\beta = 0$, we see that the equation for $\hat{w}(y)$ is uncoupled from the rest of the equations and $\hat{u}(y)$ and $\hat{v}(y)$ can be solved independently of $\hat{w}(y)$. As the direction of the oblique wave is changed, the shape of the velocity profile $\overline{u}(y)$ remains unchanged. However, the magnitude is given by the projection of the actual velocity vector on that direction. Thus the wave propogation in the direction of the velocity vector $\overline{U}(y)$ corresponds to the highest effective velocity or Reynolds number.

Usually a laminar flow is always stable at sufficiently low Reynolds numbers because of viscous dissipation. A finite critical Reynolds number may be reached when at least some self-sustained oscillations become possible. In the present case, it then follows that, the critical Reynolds number for an oblique wave must be higher than that for the wave propagating in the direction of the steady mean flow. Because of the uncoupling, one needs to consider two dimensional disturbances only. This is a very important result, as it justifies the consideration of two-dimensional disturbances only. The uncoupling of the lateral oscillations is possible only when the curvature of the surface is negligible. The stability boundary for the

oblique wave is obtained by shifting the stability boundary for the two-dimensional disturbances towards higher Reynolds numbers at each wave number α. As a result, however, certain combinations of α and R may become unstable for the oblique wave although stable for the two dimensional disturbances.

3.4 STABILITY OF PARALLEL FLOWS WITH RESPECT TO LONGITUDINAL-VORTEX DISTURBANCES AND TRANSVERSE-VORTEX DISTURBANCES:

Disregarding the lateral disturbances w', the disturbances u', v' may be visualised as being equivalent to a disturbance vorticity distribution which is periodic in x and is in the direction of the z axis only. We may refer to this type of disturbances as 'Transverse-vortex disturbances'.

On the other hand, suppose we consider the alternative extreme of setting $\alpha = 0$. Then the roles of u' and w' are exchanged (while dealing in cartesian coordinates only) and nothing new follows. However, the disturbances v' and w' are solved independently of u'. These disturbances may be defined as 'Longitudinal-vortex disturbances', the vorticity component being parallel to the x axis. Although the longitudinal-vortex disturbance is of trivial importance for parallel flows, over a flat surface with steady state velocity in the x direction only, it turns out to be of

great importance in case there is a centrifugal force due to the curvature of the surface in the x direction or in case other body forces such those due to thermal buoyancy exist.

3.5 ANALYSIS OF TWO DIMENSIONAL DISTURBANCES:

For a parallel flow with velocity $\bar{u}$ (y) under transverse-vortex disturbances, it is only necessary to consider two dimensional disturbances u' and v'. The equations of motion are obtained from equations (3.3.2) and (3.3.3) by putting $\beta = 0$ and $w' = 0$. The operators given by equation (3.3.3) reduce to,

$$D \equiv d/dy$$
$$L = (D^2 - \alpha^2) \qquad (3.5.1)$$
$$\bar{\bar{L}} = L(1-L/K)$$

and the equations (3.3.2) reduce to,

$$\left[\frac{\bar{\bar{L}}}{R} - i\alpha(\bar{u}-c)\right]\hat{u} = (D\bar{u})\hat{v} + i\alpha\hat{p}$$
$$\left[\frac{\bar{\bar{L}}}{R} - i\alpha(\bar{u} - c)\right]\hat{v} = D\hat{p}$$

By eliminating the pressure term and by making use of the continuity equation, the above system of equations reduces to,

$$\frac{\bar{\bar{L}}}{R}\left[L\hat{v}\right] = i\alpha R\left[(\bar{u}-c)\,L - D^2\bar{u}\right]\hat{v} \qquad (3.5.2)$$

Subject to boundary conditions:

$$\hat{u} = \hat{v} = n_j\,M'_{ji} = 0 \quad \text{at } y = y_1 \text{ and } y_2.$$

In the limiting case of vanishing couple stresses the non-dimensional parameter $K \to \infty$; so that equation (3.5.2) takes the form,

$$(D^2 - \alpha^2)^2 \hat{v} = i\alpha R \left[(\bar{u} - c)(D^2 - \alpha^2) \hat{v} - (D^2 \bar{u}) \hat{v} \right] \tag{3.5.3}$$

Subject to the boundary conditions

$$\hat{u} = \hat{v} = 0 \quad \text{at } y = y_1 \text{ and } y_2$$

Equation (3.5.3) is the 'Orr-Sommerfeld' equation.

Equation (3.5.2) is essentially an equation for vorticity. A general two dimensional motion can be specified by a stream function

$\psi(x, y, t)$ such that,

$$u = \partial \psi / \partial y$$
$$v = - \partial \psi / \partial x \tag{3.5.4}$$

The vorticity is then given by,

$$\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$
$$= \Delta \psi$$

where $\Delta \equiv \partial^2 / \partial x^2 + \partial^2 / \partial y^2$

In terms of the vorticity ζ, equations (2.5.8) reduce to,

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \zeta}{\partial x} - \frac{\partial \psi}{\partial x} \left(\frac{\partial \zeta}{\partial y} \right) = \frac{1}{R} \Delta \zeta - \frac{1}{RK} \Delta \Delta \zeta \tag{3.5.5}$$

To study the effects of small disturbances on the steady flow given by $\bar{\Psi}(x,y)$, we make the substitutions

$$\Psi(x,y,t) = \bar{\Psi}(x,y) + \psi'(x,y,t)$$

and

$$\zeta(x,y,t) = \bar{\zeta}(x,y) + \zeta'(x,y,t) \tag{3.5.6}$$

in equation (3.5.5). Linearising the resulting equations, we get,

$$\frac{\partial \zeta'}{\partial t} + \bar{u}\frac{\partial \zeta'}{\partial x} - \frac{\partial \psi'}{\partial x}\frac{\partial \bar{\zeta}}{\partial y} = \frac{1}{R}\Delta\zeta' - \frac{1}{RK}\Delta\Delta\zeta' \tag{3.5.7}$$

For parallel flows $\bar{\Psi}(x,y)$ is a cubic in y and is independent of x.

The differentialequation for ψ' is linear and has coefficients independent of x and t. Consequently, we may expect a solution of the form,

$$\psi'(x,y,t) = \phi(y)\, e^{i\alpha(x-ct)} \tag{3.5.8}$$

The disturbance vorticity is then obtained as,

$$\zeta' = (D^2-\alpha^2)\,\phi(y)\, e^{i\alpha(x-ct)}$$

and equation (3.5.7) takes the form

$$(D^2-\alpha^2)^2\,(1-(D^2-\alpha^2)/K)\,\phi = i\alpha R\left[(\bar{u}-c)(D^2-\alpha^2)\,\phi - (D^2\,\bar{u})\,\phi\right] \tag{3.5.9}$$

The appropriate boundary conditions are

$$\begin{aligned} u' &= \phi = 0 \\ v' &= D\phi = 0 \qquad \text{at } y = y_1 \text{ and } y_2 \\ M'_{yz} &= (D^2-\alpha^2)D\phi = 0 \end{aligned}$$

The amplitude S of the vorticity function is given by,

$$S = (D^2 - \alpha^2)\,\phi \qquad (3.5.10)$$

Therefore, the boundary conditions can be written as

$$\begin{aligned} \phi &= 0 \\ D\phi &= 0 \qquad \text{at } y = y_1 \text{ and } y_2 \\ DS &= 0 \end{aligned} \qquad (3.5.11)$$

The equation of motion (3.5.9) subject to boundary conditions given by equations (3.5.11) results in a characteristic value problem of the type

$$F(\alpha, R, K, c) = 0 \qquad (3.5.12)$$

For each combination of α, R and K there is a characteristic value c which is in general complex. For fixed values of K, the condition $c_i = 0$ leads to a relation between α and R, that is, to a curve in the α, R plane. This curve is usually referred to as the neutral stability curve.

The stability of plane Poiseuille flow will be discussed in the next section.

4. STABILITY OF PLANE POISEUILLE FLOW

4.1 REVIEW OF LITERATURE ON STABILITY OF PLANE POISEUILLE FLOW :

The class of strictly parallel flows is limited, since $\bar{u}(y)$ can at most be quadratic in y. This includes, however, two important special cases :

(1) Plane Couette flow ;

$$\bar{u}(y) = y \quad (-1 \leqslant y \leqslant 1)$$

where V^* = The velocity of the upper plate

L^* = Half the channel width

(2) Plane Poiseuille flow ;

$$\bar{u}(y) = 1 - y^2 \quad (-1 \leqslant y \leqslant 1)$$

where V^* = The maximum velocity at the centre of the channel

L^* = Half the channel width

In plane Couette flow, the fluid motion is a simple shear caused by the relative motion of the two plates. This is obviously the simplest laminar motion conceivable, and one might expect a simple stability problem. However, no conclusive answer has yet been arrived at for this problem. All existing investigations tend to show that the flow is stable. In the case of plane Poiseuille flow an exact solution of the problem, even in a formal sense, is not possible. This flow

therefore provides one of the basic problems to be investigated for stability.

Historically the study of laminar flows, such as classical plane Couette flow and plain Poiseuille flow was started with the consideration of a transverse-vortex disturbance by Rayleigh[23] followed notably by Heisenberg[24], Tollimien[25] and Lin[26].

The stability of plane Poiseuille flow with respect to two-dimensional disturbances has been studied by many authors. A considerable amount of controversy exists because of the contradictory conclusions which have been arrived at. Heisenberg[24] concluded that plane Poiseuille flow becomes unstable at a sufficiently high Reynolds number, but did not arrive at a critical value beyond which instability begins. Lin[1] obtained the minimum critical Reynolds number of 5300, based on the maximum velocity at the centre of the channel and its half width. Calculations of curves of constant c_i were made by Shen[27] by a perturbation from Lin's neutral curve. Both Heisenberg and Lin used asymptotic series. Meksyn[28](1946) used a similar method and obtained a minimum critical Reynolds number of 6800 after certain approximations. A different method was used by Pekeris[29], who concluded that plane Poiseuille flow is stable at all Reynolds numbers. It is

believed that this conclusion arises from the fact that Pekeris's series is essentially valid only for stable cases. The disagreement of Pekeris's result with Lin's calculations led Von Neumann to suggest a direct numerical calculation which was made by Thomas[30]. In 1953, Thomas, using the Finite-difference technique, arrived at the minimum critical Reynolds number of 5780. The Orr-Sommerfeld equation of hydrodynamic stability theory has presented a challenge to both asymptotic and numerical analyses for almost half a century. In spite of its appearance as a simple linear eigenvalue problem, its structure has required development of special asymptotic methods as well as numerical techniques. The asymptotic methods are described in detail by Lin[1]. The asymptotic theory of the Orr-Sommerfeld equation has been greatly advanced in recent years. However, it is incomplete in many respects. In much of the recent work on this aspect of the problem (Lin and Rabenstein[31](1960), Sibuya[32](1963)), the major emphasis has been on the uniformity of the approximations achieved. It has been shown that uniform approximations to the solution of the Orr-Sommerfeld equation can be obtained in terms of the solutions of a simpler comparison equation. This equation, however, is also of the fourth order. All of this work has been restricted to flows with one critical point for which

the comparison equation can be solved by the method of Laplace integrals. Even for such flows, the theory is quite complicated and its extension, for example, to flows with more than one critical point, though perhaps not formally difficult, would clearly lead to further complications. In 1966, Graebel[33] applied the technique of matched expansions to the classical inner and outer solutions to obtain eigenvalue relations. The technique of Multiple-scale was used by Kuentam[34] to obtain the asymptotic solution of the Orr-Sommerfeld equation. The stability of steady and time dependent plane Poiseuille flow has been studied by Grosch[35] by making use of the method of expansions in a set of orthogonal functions. Much work has also been done in obtaining numerical solutions of the stability problem. The direct numerical calculations made by Thomas were quite lengthy and, according to Lin[1], the limited amount of work required two weeks of machine time on a high speed electronic calculator. Nachtsheim[36] successfully applied the initial value technique in solving the stability of plane Poiseuille flow. Nachtsheim obtained the minimum critical Reynolds number of 5769. The agreement of the results of Nachtsheim with those of Thomas is very good. Kaplan[37](1964) also applied the Finite-difference technique for calculating the eigenvalues. Lee and Reynolds[38] suggested two

numerical techniques to solve the Orr-Sommerfeld equation. The first is a variational approach which allows the eigenvalue problem to be reduced to an algebric problem of matrix eigenvalue determination. The second scheme involves numerical integration of the differential equations. This scheme is based on the method suggested by Kaplan with some modifications. Potter[39], making use of Reynolds technique, studied the finite amplitude stability of parallel flows.

In the following sections, the numerical scheme for solving the stability of plane Poiseuille flow with couple stresses has been developed. The method suggested by Nachtsheim is the basis of the present work.

4.2 FORMULATION OF THE PROBLEM :

Consider the flow due to a pressure gradient between two parallel plates at a distance 2h. Let the x_1 axis coincides with the lower plate. The axis x_2 is normal to x_1 axis and the x_3 axis represents the lateral direction. The governing equation of motion in a dimensional form is,

$$\rho \dot{u}_i = - p_{,i} + \mu u_{i,rr} - \eta u_{i,rrss} \qquad (4.2.1)$$

Body forces and body moments are assumed to be absent. For steady one dimensional flow of an incompressible fluid between two parallel plates, the velocity field is given by

$$u_1 = u_1(x_2)$$

$$u_2 = 0$$

$$u_3 = 0$$

Equation (4.2.1) then reduces to,

$$\frac{d^4u_1}{dx_2{}^4} - \frac{1}{\ell^2} \frac{d^2u_1}{dx_2{}^2} = -\frac{1}{\eta} \frac{dp}{dx_1} \tag{4.2.2}$$

where $p = p(x_1)$

$\ell^2 = \eta/\mu$, ℓ being a material constant.

The general solution of equation (4.2.2) is

$$u_1 = A_o + A_1 x_2 + \frac{1}{2\mu} \left(\frac{\partial p}{\partial x_1}\right) x_2^2 + B_1 \cosh(x_2/\ell) + B_2 \sinh(x_2/\ell) \tag{4.2.3}$$

The only non-zero component of the vorticity, defined by $\omega_i = 1/2\ e_{irs}\ u_{s,r}$, is given by

$$\omega_3 = -\frac{1}{2} \frac{\partial u_1}{\partial x_2}$$

the couple stress tensor is given by

$$M_{ij} = 4\eta K_{ij} + 4\eta' K_{ji}$$

where $K_{ij} = \omega_{j,i}$

the only non-zero component of interest is,

$$M_{23} = 4\eta K_{23}$$

$$= -2\eta\left[\frac{1}{\mu}\left(\frac{\partial p}{\partial x_1}\right) + \frac{B_1}{\ell^2}\cosh(x_2/\ell) - \left(\frac{B_2}{\ell^2}\right)\sinh(x_2/\ell)\right] \quad (4.2.4)$$

besides this, M_{32} is also non-zero.

The appropriate boundary conditions are,

$$\left.\begin{aligned} u_1 &= 0 \\ M_{23} &= 0 \end{aligned}\right\} \text{ at } x_2 = 0 \text{ and } x_2 = 2h \quad (4.2.5)$$

Solving equations (4.2.3) and (4.2.4) with boundary conditions given by equation (4.2.5), we obtain,

$$u_1(x_2) = \frac{h^2}{2\mu}\left(\frac{\partial p}{\partial x_1}\right)\left[-2(x_2/h) + (x_2/h)^2 + \frac{2}{a^2}\left\{1 - \cosh(x_2/\ell) + \tanh a.\ \sinh\left(\frac{x_2}{\ell}\right)\right\}\right] \quad (4.2.6)$$

The maximum velocity occurs at the centre of the channel, $x_2 = h$ and is given by,

$$u_{1max} = -\frac{h^2}{2\mu}\left(\frac{\partial p}{\partial x_1}\right)\left[1 - \frac{2}{a^2}(1 - 1/\cosh a)\right]$$

where $a = h/\ell$ (4.2.7)

Equation (4.2.6) is non-dimensionalised by taking u_{1max} as the characteristic velocity and half the channel width as the characteristic length. The non-dimensional quantities come out to be,

$$\begin{aligned} y &= x_2/h \\ u(y) &= u_1(x_2)/u_{1max} \end{aligned} \quad (4.2.8)$$

$$\bar{u}(y) = \frac{2y - y^2 - \frac{2}{a^2}\,(1 - \cosh a\,y + \tanh a\,\sinh a\,y)}{\left[1 - \frac{2}{a^2}\,(1 - 1/\cosh a)\right]} \qquad (4.2.9)$$

The non-dimensional parameter K takes the form

$$K = L^{*2}/\ell^2$$

$$= h^2/\ell^2 = a^2 \qquad (4.2.10)$$

The governing equation for the stability of plane Poiseuille flow, subject to two dimensional disturbances, is obtained from equations (3.5.9) by substituting for values of K and $\bar{u}$ as,

$$(D^2 - \alpha^2)^2 \left[1 - \frac{(D^2 - \alpha^2)}{a^2}\right] \phi = i\alpha R \left\{ (\bar{u} - c)\,(D^2 - \alpha^2)\,\phi - (D^2\,\bar{u})\,\phi \right\} \qquad (4.2.11)$$

The boundary conditions are,

$$\phi = 0$$

$$D\phi = 0 \qquad \text{at } y = 0 \text{ and } y = 2 \qquad (4.2.12)$$

$$DS = 0$$

where $S = (D^2 - \alpha^2)\,\phi$

Our task is to obtain a numerical solution of equation (4.2.11) with the boundary conditions (4.2.12). We are mainly interested in finding out the eigenvalues c and eigenfunctions ϕ for a particular set of parameters R, α and a. A discussion of the general nature of the numerical solution of equation (4.2.11) is given in the next section.

4.3 NUMERICAL SOLUTION OF THE DIFFERENTIAL EQUATION:

The disturbance equation with couple stresses is given by equation (4.2.11). The parameter a comes in due to the effect of couple stresses. The case of a tending to infinity corresponds to the classical problem, and the resulting equation is the well known Orr-Sommerfeld equation :

$$(D^2 - \alpha^2)^2 \phi = i\alpha R \left[(\bar{u} - c)(D^2 - \alpha^2)\phi - (D^2 \bar{u})\phi \right] \tag{4.3.1}$$

Subject to the boundary conditions,

$$\left.\begin{array}{l} \phi = 0 \\ D\phi = 0 \end{array}\right. \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 2. \tag{4.3.2}$$

The numerical solution of the Orr-Sommerfeld equation has been given by many authors. A brief description of the numerical solution of equation (4.3.1) will help in understanding the nature of equation (4.2.11). Hence, we will start our discussion with a review of the numerical solution of the Orr-Sommerfeld equation.

In solving the Orr-Sommerfeld equation, one is faced with the task of solving a linear ordinary differential equation of the form

$$L_2 \phi + R^{-1} L_4 \phi = 0 \tag{4.3.3}$$

where L_4 is a fourth order operator and L_2 is a second order operator. If R were of order unity there would have been no real problem. However, in the present problem R is of the

order of 10^4, and hence the equation is singular in nature. Since $\bar{u}$ (y) is symmetrical about the centre line, it is advisable to consider the solution from the centre to one of the walls. The boundary conditions can then be specified at the centre and at the wall. There will be two linearly independent solutions of the equation satisfying the two central boundary conditions. An appropriate linear combination of these solutions must be obtained to satisfy the two wall conditions. It is known from the asymptotic theory (Lin[1]) that one of the solutions grows rapidly away from the centre, a numerical solution of this function (Reynolds[38]) for R = 10000 (corresponding to Thomas's tabulated results) showed a growth of the order of 10^{18} from the centre to the wall. It is clear that a numerical generation of the second solution, which is linearly independent, will be difficult. Any slight round-off error will in effect throw in some small multiple of the growing solution, which is likely to dominate the second solution by the time the wall is reached. This was indeed observed in an experiment (Reynolds[38]) in which an eigenfunction calculation was started at the centre with the inviscid starting conditions. By the time the wall was reached the solution was, to eight digits, a multiple of the growing solution. In fact, it was of the order of 10^{10},

suggesting that it was present initially to the order of one part in 10^8, the maximum accuracy of single precision on the machine. Of course, the final solution does not exhibit this rapid growth, which means that only a very small amount of the growing solution is required. The problem then is to generate numerically a second solution which is not merely a multiple of the growing solution. There are essentially two approaches. First, one might use multiple precision, extending the accuracy of the digital computations to, say, 16 digits. The solution can then be carried out in two parts, inwards from the centre and outwards from the wall and matched somewhere in between. This may limit the growth of the error to one part in 10^8 on either side, for which 16 digit computations might suffice. A scheme of this type was used by Nachtsheim. A second scheme used by Kaplan (1964) involves a suppression of the growing solution during the calculation of the well behaved solution. A general description of this technique can be found in Reynolds [38] paper. Kaplan's method has been used in a variety of recent calculations with remarkable success. Experience has shown that the initial guess must be relatively good for fast convergence.

The finite difference technique suggested by Thomas and used by Kaplan is very time-consuming and complicated. Moreover, to start the numerical integration, the initial values at considerably more points should be known. Keeping these points in view, the method suggested by Nachtsheim has been used in solving equation (4.2.11). The technique will be discussed in detail in the following sections.

4.40 METHOD OF SOLUTION : INITIAL VALUE TECHNIQUE

The disturbance equation (4.2.11) is

$$(D^2-\alpha^2)^2\left[1-\frac{1}{a^2}(D^2-\alpha^2)\right]\phi = i\alpha R\left\{(\bar{u}-c)(D^2-\alpha^2)\phi-(D^2\bar{u})\phi\right\}$$

with the boundary conditions,

$$\phi = 0$$

$$D\phi = 0 \qquad \text{at } y = 0 \text{ and } y = 2.$$

$$DS = 0$$

The amplitude of the disturbance vorticity function is defined as

$$S = (D^2-\alpha^2)\phi$$

If we introduce

$$A = (D^2-\alpha^2)S$$

then, equation (4.2.11) takes the form,

$$A'' - \frac{1}{a^2}(D^2-\alpha^2)A = i\alpha R\left\{(\bar{u}-c)S-D^2\bar{u}\phi\right\}$$

Thus, the sixth-order differential equation can be reduced to a system of three second order differential equations,

$$\begin{aligned} \phi'' &= \alpha^2 \phi + S \\ S'' &= \alpha^2 S + A \\ A'' &= (a^2 + \alpha^2)A - i\alpha R a^2 \left[(\bar{u} - c) S - \bar{u}''\phi \right] \end{aligned} \tag{4.4.1}$$

The corresponding boundary conditions are,

$$\left.\begin{aligned} \phi &= 0 \\ \phi' &= 0 \\ S' &= 0 \end{aligned}\right\} \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 2 \tag{4.4.2}$$

where ' denotes differentiation with respect to y.

The solution of the disturbance equation (4.4.1) for ϕ, for given values of α , R and a, can be made to satisfy the boundary conditions (4.4.2) only for the eigenvalues of c. We are interested in finding the eigenvalues and eigen-solutions for a given set of flow parameters. It is also of interest to determine the minimum critical Reynolds number.

With regard to equation (4.4.1), of primary interest are the solutions that are even functions of y about the line y = 1. Since the velocity profile is an even function of y about y = 1, the disturbances can be separated into even and odd parts. The former, which has a simple flow pattern, usually give a lower critical Reynolds number, hence the second set of boundary conditions (at y = 2) are replaced by the conditions at centre, as

$$\begin{aligned} \phi' &= 0 \\ \phi''' &= 0 \qquad \text{at } y = 1 \\ S' &= 0 \end{aligned} \qquad (4.4.3)$$

Thus our characteristic value problem may be treated in the interval (0,1), in which $\bar{u}$ (y) is increasing monoterically.

The basic idea is to solve the differential equation for assumed values of the eigenvalue and other initial values and to determine the appropriate changes in these values so that the boundary conditions are ultimately satisfied.

Equation (4.4.1) basically has six linear independent solutions, some of which grow exponentially at a rapid rate. The difficulty is overcome by using Double-precision arithmetic (16 significant figures) and carrying out the integration of equation (4.4.1) in two parts, inwards from the centre, $y = 1$, and outwards from the wall at $y = 0$ and then matching the two solutions in the middle at $y_c = 5$.

In each case, the integration is performed in the direction in which the wanted function is increasing and the unwanted function is decreasing. It is important to include as much information as possible about the wanted solution in the statement of the problem.

In general the most important criteria are the proper choices of y_c and Y (which in this case is Y = 1). A poor choice of y_c could lead to convergence to a different solution. The best choice of y_c is that value of y for which, for the required eigenvalue, the coefficients of the dominating functions vanish in the equation which involves the derivatives of these dominating functions. If the chosen y_c is somewhere in between the critical values corresponding to two eigenvalues, we may converge to either of these eigenvalues. Indications are that we would converge to the root closest to the initial guess but we may loose the desired speed of quadratic convergence.

A poor choice of Y has two effects. The first is that if Y is too small to represent infinitv for the eigen-solution sought, we may converge to a different result or may not converge at all. The second effect is that, if Y is much larger than necessary, the solution will converge but at a linear rather than at a quadratic rate. This is probably due to the fact that the best matching point is not the same for all functions and if we necessarily cover too large a range there is a significant build up of errors which inhibits the process. The shape of the eigenfunction $\phi(y)$, but not the amplitude, is fixed by equation (4.2.11). There

are two approaches for fixing the amplitude of the eigenfunction ϕ (y).

Firstly, if we define the amplitude of the velocity distribution in some particular manner, this definition will fix the amplitude of ϕ, apart from a constant of modulus unity. An alternative approach is to normalise ϕ in some manner. This will in turn define the amplitude of the velocity of the disturbances. In plane Poiseuille flow the latter choice is more convenient. Following the normalisation suggested by Nachtsheim we set,

$$\phi(1) = 1 \quad \text{for even modes}$$

$$\phi'(1) = 1 \quad \text{for odd modes}$$

As we are interested in even modes, we take the normalising condition $\phi(1) = 1$. This fixes the size of the whole solution. Then our task is to solve equation (4.4.1) with the boundary conditions given by equation (4.4.2). As discussed earlier, the initial values at y = 1 and y = 0 are specified in the following manner.

For the forward solution, starting from y = 1, the initial conditions are,

$$\begin{aligned}
\phi_f &= 1 \\
\phi_f' &= 0 \\
S_f &= p \\
S_f' &= 0 \\
A_f &= q \\
A_f' &= 0
\end{aligned} \tag{4.4.4}$$

The backward solution is started at y = 0 with the initial conditions,

$$\begin{aligned}
\phi_b &= 0 \\
\phi_b' &= 0 \\
S_b &= r \\
S_b' &= 0 \\
A_b &= g \\
A_b' &= t
\end{aligned} \tag{4.4.5}$$

here the suffixes f and b represent the forward and the backward solution respectively.

The variation parameters p and q in the forward solution and r, g and t in the backward solution cannot be fixed arbitrarily but must be determined in the iterative process whilst attempting to match the solutions at $y_c = .5$. The solutions must be continuous. Matching at $y = y_c$ requires that

$$
\begin{aligned}
\phi_f &= \phi_b \\
\phi_f' &= \phi_b' \\
S_f &= S_b \\
S_f' &= S_b' \\
A_f &= A_b \\
A_f' &= A_b'
\end{aligned}
\qquad (4.4.6)
$$

If these conditions are satisfied, all the higher derivatives agree and matching is accomplished.

The quantities $\phi_f(y_c)$, $\phi_f'(y_c)$, $S_f(y_c)$, $S_f'(y_c)$, $A_f(y_c)$ and $A_f'(y_c)$ are functions of p, q and c and the quantities $\phi_b(y_c)$, $\phi_b'(y_c)$, $S_b(y_c)$, $S_b'(y_b)$, $A_b(y_b)$ and $A_b'(y_b)$ are functions of r, g, t and c. The quantities ϕ, S, A, p, q, r, g, t and c are complex. Successive changes are made in the first estimates of the variational parameters so that equations (4.4.6) are ultimately satisfied.

The Newton-Raphson method is used to fulfil the conditions imposed by equations (4.4.6). If the chosen values of p, q, r, g, t and c produce a solution that approximately satisfies equation (4.4.6), a better approximation is obtained by starting with $p + \Delta p$, $q + \Delta q$, $r + \Delta r$, $g + \Delta g$, $t + \Delta t$ and $c + \Delta c$. The increments Δp, Δq, Δr, Δg, Δt and Δc are solutions of the equations

$$\begin{aligned}
&\phi_f - \phi_b + \Delta p \frac{\partial}{\partial p}(\phi_f - \phi_b) + \Delta q \frac{\partial}{\partial q}(\phi_f - \phi_b) + \Delta c \frac{\partial}{\partial c}(\phi_f - \phi_b) \\
&\qquad + \Delta r \frac{\partial}{\partial r}(\phi_f - \phi_b) + \Delta g \frac{\partial}{\partial g}(\phi_f - \phi_b) + \Delta t \frac{\partial}{\partial t}(\phi_f - \phi_b) = 0 \\
&\phi_f' - \phi_b' + \Delta p \frac{\partial}{\partial p}(\phi_f' - \phi_b') + \Delta q \frac{\partial}{\partial q}(\phi_f' - \phi_b') + \Delta c \frac{\partial}{\partial c}(\phi_f' - \phi_b') \\
&\qquad + \Delta r \frac{\partial}{\partial r}(\phi_f' - \phi_b') + \Delta g \frac{\partial}{\partial g}(\phi_f' - \phi_b') + \Delta t \frac{\partial}{\partial t}(\phi_f' - \phi_b') = 0 \\
&S_f - S_b + \Delta p \frac{\partial}{\partial p}(S_f - S_b) + \Delta q \frac{\partial}{\partial q}(S_f - S_b) + \Delta c \frac{\partial}{\partial c}(S_f - S_b) \\
&\qquad + \Delta r \frac{\partial}{\partial r}(S_f - S_b) + \Delta g \frac{\partial}{\partial g}(S_f - S_b) + \Delta t \frac{\partial}{\partial t}(S_f - S_b) = 0 \\
&S_f' - S_b' + \Delta p \frac{\partial}{\partial p}(S_f' - S_b') + \Delta q \frac{\partial}{\partial q}(S_f' - S_b') + \Delta c \frac{\partial}{\partial c}(S_f' - S_b') \\
&\qquad + \Delta r \frac{\partial}{\partial r}(S_f' - S_b') + \Delta g \frac{\partial}{\partial g}(S_f' - S_b') + \Delta t \frac{\partial}{\partial t}(S_f' - S_b') = 0 \\
&A_f - A_b + \Delta p \frac{\partial}{\partial p}(A_f - A_b) + \Delta q \frac{\partial}{\partial q}(A_f - A_b) + \Delta c \frac{\partial}{\partial c}(A_f - A_b) \\
&\qquad + \Delta r \frac{\partial}{\partial r}(A_f - A_b) + \Delta g \frac{\partial}{\partial g}(A_f - A_b) + \Delta t \frac{\partial}{\partial t}(A_f - A_b) = 0 \\
&A_f' - A_b' + \Delta p \frac{\partial}{\partial p}(A_f' - A_b') + \Delta q \frac{\partial}{\partial q}(A_f' - A_b') + \Delta c \frac{\partial}{\partial c}(A_f' - A_b') \\
&\qquad + \Delta r \frac{\partial}{\partial r}(A_f' - A_b') + \Delta g \frac{\partial}{\partial g}(A_f' - A_b') + \Delta t \frac{\partial}{\partial t}(A_f' - A_b') = 0
\end{aligned} \tag{4.4.7}$$

in which the functions and the partial derivatives that constitute the coefficients are evaluated at y_c.

The partial derivatives are obtained by solving initial-value problems. These equations are obtained by partial differentiation of the terms in equations (4.4.1).

The coefficients of equations (4.4.1) are analytic functions of y and the parameters α, R, a and c. The solutions of equation (4.4.1) therefore have the same analytic properties and posses the required partial derivatives.

The quantities $\frac{\partial \phi_f}{\partial p} = \phi_{f,p}$, $\frac{\partial S_f}{\partial p} = S_{f,p}$ and $\frac{\partial A_f}{\partial p} = A_{f,p}$ for the forward solution satisfy the system of equations

$$
\begin{aligned}
\phi''_{f,p} &= \alpha^2 \phi_{f,p} + S_{f,p} \\
S''_{f,p} &= \alpha^2 S_{f,p} + A_{f,p} \\
A''_{f,p} &= (\alpha^2 + a^2) A_{f,p} - i\alpha Ra^2 \left[(\bar{u} - c) S_{f,p} - \bar{u}'' \phi_{f,p} \right]
\end{aligned}
\tag{4.4.8}
$$

with the initial conditions at y = 1 given by,

$$
\begin{aligned}
\phi_{f,p} &= 0 \\
\phi'_{f,p} &= 0 \\
S_{f,p} &= 1 \\
S'_{f,p} &= 0 \\
A_{f,p} &= 0 \\
A'_{f,p} &= 0
\end{aligned}
\tag{4.4.9}
$$

Similar equations can be written for the variational parameter q by replacing p by q in equations (4.4.8), and by satisfying the following initial conditions at y = 1 :

$$\begin{aligned}
\phi_{f,q} &= 0 \\
\phi'_{f,q} &= 0 \\
S_{f,q} &= 0 \\
S'_{f,q} &= 0 \\
A_{f,q} &= 1 \\
A'_{f,q} &= 0
\end{aligned} \tag{4.4.10}$$

The variational equations for c are given by,

$$\begin{aligned}
\phi''_{f,c} &= \alpha^2 \phi_{f,c} + S_{f,c} \\
S''_{f,c} &= \alpha^2 S_{f,c} + A_{f,c} \\
A''_{f,c} &= (a^2 + \alpha^2) A_{f,c} - i\alpha R a^2 \left[(\bar{u} - c) S_{f,c} - \bar{u}'' \phi_{f,c} \right]
\end{aligned} \tag{4.4.11}$$

With initial conditions at y = 1 given by,

$$\begin{aligned}
\phi_{f,c} &= 0 \\
\phi'_{f,c} &= 0 \\
S_{f,c} &= 0 \\
S'_{f,c} &= 0 \\
A_{f,c} &= 0 \\
A'_{f,c} &= 0
\end{aligned} \tag{4.4.12}$$

For the backward solution, the variational equations for r, g and t are the same as those for p in the forward solution, except that the initial conditions at y = 0 in this case are

$$
\begin{aligned}
\phi_{b,r} &= 0 \\
\phi'_{b,r} &= 0 \\
S_{b,r} &= 1 \\
S'_{b,r} &= 0 \\
A_{p,r} &= 0 \\
A'_{p,r} &= 0
\end{aligned}
\tag{4.4.13}
$$

$$
\begin{aligned}
\phi_{b,g} &= 0 \\
\phi'_{b,g} &= 0 \\
S_{b,g} &= 0 \\
S'_{b,g} &= 0 \\
A_{b,g} &= 0 \\
A'_{b,g} &= 0
\end{aligned}
\tag{4.4.14}
$$

for t and c the initial conditions at $y = 0$ are

$$
\begin{aligned}
\phi_{b,t} &= 0 \\
\phi'_{b,t} &= 0 \\
S_{b,t} &= 0 \\
S'_{b,t} &= 0 \\
A_{b,t} &= 0 \\
A'_{b,t} &= 1
\end{aligned}
\tag{4.4.15}
$$

For c, the initial conditions at $y = 0$ are

$$\begin{aligned}
\phi_{b,c} &= 0 \\
\phi'_{b,c} &= 0 \\
S_{b,c} &= 0 \\
S'_{b,c} &= 0 \\
A_{b,c} &= 0 \\
A'_{b,c} &= 0
\end{aligned} \tag{4.4.16}$$

The quantities in the forward solution are functions of y, p, q and c only. Similarly in the backward solutions the quantities are dependent on y, r, g, t and c only. Hence the partial derivatives with respect to r, g and t in the forward solution and with respect to p and q in the backward solution are zero.

That is,

$$\begin{aligned}
\phi_{f,r} = \phi_{f,g} = \phi_{f,t} = \phi_{b,p} = \phi_{b,q} &= 0 \\
\phi'_{f,r} = \phi'_{f,g} = \phi'_{f,t} = \phi'_{b,p} = \phi'_{b,q} &= 0 \\
S_{f,r} = S_{f,g} = S_{f,t} = S_{b,p} = S_{b,q} &= 0 \\
S'_{f,r} = S'_{f,g} = S'_{f,t} = S'_{b,p} = S'_{b,q} &= 0 \\
A_{f,r} = A_{f,g} = A_{f,t} = A_{b,p} = A_{b,q} &= 0 \\
A'_{f,r} = A'_{f,g} = A'_{f,t} = A'_{b,p} = A'_{b,q} &= 0
\end{aligned} \tag{4.4.17}$$

Equations (4.4.7) then reduce to the following set of equations,

$$
\begin{aligned}
&\phi_f - \phi_b + \Delta p\, \phi_{f,p} + \Delta q\, \phi_{f,q} + \Delta c\, (\phi_{f,c} - \phi_{b,c}) - \Delta r\, \phi_{b,r} \\
&\qquad\qquad - \Delta g\, \phi_{b,g} - \Delta t\, \phi_{b,t} = 0 \\
&\phi'_f - \phi'_b + \Delta p\, \phi'_{f,p} + \Delta q\, \phi'_{f,q} + \Delta c\, (\phi'_{f,c} - \phi'_{b,c}) - \Delta r\, \phi'_{b,r} \\
&\qquad\qquad - \Delta g\, \phi'_{b,g} - \Delta t\, \phi'_{b,t} = 0 \\
&S_f - S_b + \Delta p\, S_{f,p} + \Delta q\, S_{f,q} + \Delta c\, (S_{f,c} - S_{b,c}) - \Delta r\, S_{b,r} \\
&\qquad\qquad - \Delta g\, S_{b,g} - \Delta t\, S_{b,t} = 0 \qquad\qquad (4.4.18) \\
&S'_f - S'_b + \Delta p\, S'_{f,p} + \Delta q\, S'_{f,q} + \Delta c\, (S'_{f,c} - S'_{b,c}) - \Delta r\, S'_{b,r} \\
&\qquad\qquad - \Delta g\, S'_{b,g} - \Delta t\, S'_{b,t} = 0 \\
&A_f - A_b + \Delta p\, A_{f,p} + \Delta q\, A_{f,q} + \Delta c\, (A_{f,c} - A_{b,c}) - \Delta r\, A_{b,r} \\
&\qquad\qquad - \Delta g\, A_{b,g} - \Delta t\, A_{b,t} = 0 \\
&A'_f - A'_b + \Delta p\, A'_{f,p} + \Delta q\, A'_{f,q} + \Delta c\, (A'_{f,c} - A'_{b,c}) - \Delta r\, A'_{b,r} \\
&\qquad\qquad - \Delta g\, A'_{b,g} - \Delta t\, A'_{b,t} = 0
\end{aligned}
$$

Hence, there are 6 complex equations to determine the 6 complex quantities Δp, Δq, Δc, Δr, Δg and Δt.

Each step of the integration scheme is carried out by starting with the estimates for p, q, c, r, g and t. The forward system of equations (4.4.1) is then integrated, step by step, subject to the initial conditions given by

equation (4.4.4). This will necessitate the simultaneous solution of the variational equations (4.4.8) and similar equations for q and c together with the initial conditions given by equations (4.4.9), (4.4.10) and (4.4.12). The backward system is then integrated with the initial conditions given by equations (4.4.5). The four variational system of equations similar to equations (4.4.8) and (4.4.11) but with the initial conditions given by equations (4.4.13), (4.4.14), (4.4.15) and (4.4.16) are integrated simultaneously with the backward system of equations similar to equations (4.4.1). The forward and backward solutions are compared at the matching point, and the coefficients in equations(4.4.18) are evaluated. Equations (4.4.18) are then solved for Δp, Δq, Δc, Δr, Δg and Δt, and the resulting solution gives an estimate for the increments required for the next solution. This procedure is repeated till two consecutive solutions agree to a specified accuracy.

Only variations with respect to the real parts of p, q, c, r, g and t need to be obtained by this procedure. The solutions of equations (4.4.1) are analytic functions of p, q, c, r, g and t. Therefore, the real and imaginary parts of the complex derivatives, appearing in the coefficients of equations (4.4.18), can be expressed in terms of the derivatives with respect to the real quantities only.

4.5 EQUATIONS IN REAL FORM :

The differential equations, for the real quantities, are obtained by separating the original equations into real and imaginary parts. The equations (4.4.1), written in real form, come out to be,

$$\phi_r'' = \alpha^2 \phi_r + S_r$$

$$\phi_i'' = \alpha^2 \phi_i + S_i$$

$$S_r'' = \alpha^2 S_r + A_r$$

$$S_i'' = \alpha^2 S_i + A_i$$

$$A_r'' = (a^2 + \alpha^2) A_r + \alpha R a^2 \left[- S_r C_i + S_i (\bar{u} - c_r) - \bar{u}'' \phi_i \right]$$

and

$$A_i'' = (a^2 + \alpha^2) A_i + \alpha R a^2 \left[-S_i c_i - S_r (\bar{u} - c_r) + \bar{u}'' \phi_r \right] \tag{4.5.1}$$

where the suffixes r and i represent the real and imaginary parts respectively.

The real form of the variational equations (4.4.8) and and (4.4.11) comes out to be :

$$
\begin{aligned}
\phi''_{f_r,p_r} &= S_{f_r,p_r} + \alpha^2 \phi_{f_r,p_r} \\
\phi''_{f_i,p_r} &= S_{f_i,p_r} + \alpha^2 \phi_{f_i,p_r} \\
S''_{f_r,p_r} &= A_{f_r,p_r} + \alpha^2 S_{f_r,p_r} \\
S''_{f_i,p_r} &= A_{f_i,p_r} + \alpha^2 S_{f_i,p_r} \qquad (4.5.2) \\
A''_{f_r,p_r} &= (a^2 + \alpha^2) A_{f_r,p_r} + \alpha R a^2 \Big[- S_{f_r,p_r} c_i \\
&\quad + S_{f_i,p_r} (\bar{u} - c_r) - \bar{u}'' \phi_{f_i,p_r} \Big]
\end{aligned}
$$

and
$$
\begin{aligned}
A''_{f_i,p_r} &= (a^2 + \alpha^2) A_{f_i,p_r} + \alpha R a^2 \Big[- S_{f_i,p_r} c_i \\
&\quad - S_{f_r,p_r} (\bar{u} - c_r) + \bar{u}'' \phi_{f_r,p_r} \Big]
\end{aligned}
$$

The real form of the variational equations for q, r, g and t can be obtained from equations (4.5.2) by replacing p by q, r, g and t respectively. The variational equations for c, for both the forward and backward solutions, come out to be:

$$
\begin{aligned}
\phi''_{r,c_r} &= S_{r,c_r} - \alpha^2 \phi_{r,c_r} \\
\phi''_{i,c_r} &= S_{i,c_r} + \alpha^2 \phi_{i,c_r} \\
S''_{r,c_r} &= A_{r,c_r} + \alpha^2 S_{r,c_r} \\
S''_{i,c_r} &= A_{i,c_r} + \alpha^2 S_{i,c_r} \\
A''_{r,c_r} &= (a^2 + \alpha^2) A_{r,c_r} + \alpha R a^2 \Big[- S_{r,c_r} c_i \\
&\quad + S_{i,c_r} (\bar{u} - c_r) - S_i - \bar{u}'' \phi_{i,c_r} \Big]
\end{aligned}
$$

and
$$
\begin{aligned}
A''_{i,c_r} &= (a^2 + \alpha^2) A_{i,c_r} + \alpha R a^2 \Big[- S_{i,c_r} c_i \\
&\quad - S_{r,c_r} (\bar{u} - c_r) + S_r + \bar{u}'' \phi_{r,c_r} \Big]
\end{aligned}
$$

The real part of the linear equations (4.4.18) comes out to be,

$$(\emptyset_f)_r - (\emptyset_b)_r + (\emptyset_{f,p})_r \Delta p_r - (\emptyset_{f,p})_i \Delta p_i + (\emptyset_{f,q})_r \Delta q_r$$

$$- (\emptyset_{f,q})_i \Delta q_i + \boxed{(\emptyset_{f,c})_r - (\emptyset_{b,c})_r} \Delta c_r$$

$$- \boxed{(\emptyset_{f,c})_i - (\emptyset_{b,c})_i} \Delta c_i - (\emptyset_{b,r})_r \Delta r_r$$

$$+ (\emptyset_{b,r})_i \Delta r_i - (\emptyset_{b,g})_r \Delta g_r + (\emptyset_{b,g})_i \Delta g_i$$

$$- (\emptyset_{b,t})_r \Delta t_r + (\emptyset_{b,t})_i \Delta t_i = 0$$

Similarly, the other equations can also be split into their real and imaginary parts.

Since the derivatives, with respect to the real quantities, are the one that are actually calculated, it is necessary to express the real and imaginary parts of the complex derivatives in terms of the real quantities only. For example, in the case of $\emptyset_{f,p}$, $(\emptyset_{f,p})_r = (\emptyset_f)_{r,p_r}$ and $(\emptyset_{f,p})_i = (\emptyset_f)_{i,p_r}$.

Thus we have to solve twelve equations for the corrections and thirty variational equations for obtaining the desired quantities.

4.6 INTEGRATION FORMULAS :

The integration is performed by using the fifth-order Predictor Corrector method of Milne, which uses a fourth-order Runge-Kutta method to obtain starting values.

Let the system of n equations, to be solved, be given in the form

$$y_i'' = f_i\ (x,\ y_1,\ y_2\ \ldots\ldots\ldots\ y_n)\ (i = 1,\ 2\ \ldots\ldots n) \tag{4.6.1}$$

with the initial conditions

$$y_i\ (x_o) = y_{io}\ ;\quad y_i'\ (x_o) = y_{io}'\ (i = 1,\ 2\ \ldots\ldots\ n)$$

Let $y_{i,k}$ and $y_{i,k}'$ be the values of y_i and y_i' at $x = x_k$ and $f_{i,k}$ be the second derivative of y_i at $x = x_k$. h denotes the step size. The special Runge-Kutta formulas used are as follows :

$$K_{i1} = hf_i\ (x_k,\ \ y_{i,k})$$

$$K_{i2} = hf_i\ (x_k + \frac{h}{2},\ \ y_{i,k} + \frac{h}{2}\,y_{i,k}' + \frac{h}{8}\ \ K_{i1})$$

$$K_{i3} = hf_i\ (x_k + h,\ \ y_{i,k} + hy_{i,k}' + \frac{h}{2}\ \ K_{i2}) \tag{4.6.2}$$

$$y_{i,k+1} = y_{i,k} + h\left[y_{i,k}' + \frac{1}{6}\,(K_{i1} + 2K_{i2})\right]$$

and $$y_{i,k+1}' = y_{i,k}' + \frac{1}{6}\,(K_{i1} + 4K_{i2} + K_{i3})$$

where $f_i\ (x_k,\ y_{i,k})$ stands for $f_i\ (x_k,\ y_{1,k},\ y_{2,k}\ \ldots y_{n,k})$.

The Milne Predictor-Corrector formulas for solving the system, given by equation (4.6.1), are

$$p_{i,k+1} = y_{i,k} + y_{i,k-2} - y_{i,k-3} + h^2/4\ (5f_{i,k} + 2f_{i,k-1} - 5f_{i,k-2})$$

$$y_{i,k+1} = 2y_{i,k} - y_{i,k-1} + h^2/12\left[f_i(x_{k+h},\ p_{i,k+1}) + 10f_{i,k} + f_{i,k-1}\right] \tag{4.6.3}$$

The Corrector formula, given by equation (4.6.3), is applied only once, so that only two derivative evaluations are needed for each step of the Milne integration. The starting values needed in the Predictor formula (4.6.3) are obtained by using equations (4.6.2).

5. RESULTS AND CONCLUSIONS

5.1 NUMERICAL RESULTS FOR PLANE POISEUILLE FLOW:

To study the stability of plane Poiseuille flow, the eigenvalue c have been calculated for different flow parameters. Relations between various flow parameters have been shown graphically.

Couple stresses affect the steady state velocity profile. It can be seen from Fig. 1 that at any point, the steady state value of the velocity is smaller for lower values of a. This essentially means that the effect of couple stress is equivalent to an apparent increase in viscosity.

Neutral stability curves for plane Poiseuille flow for different a have been shown in Fig. 2. Part of the neutral stability curve (Lin[1]) for the nonpolar case has also been plotted for purposes of comparison. The neutral stability curve $c_i(\alpha, R, a) = 0$. separates the regions of stability and instability in the α-R plane. The lower branch of the stability curve exhibits the stabilizing nature of viscosity. On the upper branch, the effect of viscosity is to destabilize the flow. The effect of this destabilizing mechanism is to shift the phase difference between the u' and v' components of the disturbances. This helps in the development of the Reynolds stress $-\rho u'v'$, proportional to $(\phi_i\phi_r' - \phi_r\phi_i')$, which

is responsible for the energy transfer between the mean flow and the disturbances. If at any stage, the energy supplied by the Reynolds stress becomes greater than the energy dissipated by the viscosity through the diffusion of vorticity, the disturbances are amplified. For smaller Reynolds numbers the effect of viscosity is to stabilize the flow. The minimum critical Reynolds number indicates the value of the Reynolds number below which the disturbances are damped. It can be seen from Figs. 2 and 3 that the minimum critical Reynolds number first decreases and then increases with the increase in the couple stresses. Also from Figs. 2 and 4 it follows that the critical wave number increases with the increase of couple stresses (that is with decreasing a). The limiting case of zero couple stresses ($a \rightarrow \infty$) corresponds to the equation of motion (3.5.3) which governs the stability in the nonpolar case. It can therefore be assumed that the stability curve for $a \rightarrow \infty$ will be the same as the stability curve for the nonpolar case. From Fig. 2 it can be seen that, in the presence of couple stresses the domain of the unstable region increases appreciably in comparison to the nonpolar case. It should also be noted that the instability occurs at higher wave numbers in the presence of couple stresses. Table 1 gives the minimum critical Reynolds numbers and the corresponding wave numbers for different values of a as well as for the nonpolar case.

The mode shapes of the eigensolutions ϕ for the wave number $\alpha = 1$ and Reynolds number R=6000, have been shown for different values of a in Fig. 5. It can be seen that ϕ_r is on the whole much greater than ϕ_i. It can also be seen that ϕ_i is concentrated in the critical layer region (where the phase velocity c_r equals the mean velocity $\bar{u}(y)$)near the wall. The Reynolds stress and the couple stresses are also found to be concentrated in the critical layer region. When the disturbances are damped, they will vary steeply in the critical layer. This can be seen from Fig. 5 in which the mode shapes for a = 5 and a = 10 correspond to damped oscillations whereas those for a = 15 correspond to amplified oscillations. The rapid oscillations in the critical layer regions are truly remarkable and give rise to the difficulty in direct numerical integration.

Figs. 6, 7 and 8 show the variations of the eigenvalue c_i with the wave number α for different values of R and a. For sufficiently low values of R the flow remains stable. Corresponding to each Reynolds number, there are two critical wave numbers for which c_i=0. These two values define the range of instability.

The variation of c_i with R for different values of and a can be seen from Figs. 9, 10, 11, 12 and 13. For any fixed α, the critical Reynolds number corresponds to zero

value of c_i. Figs. 3 and 4 show the variation of the minimum critical Reynolds number and the minimum critical wave number with a.

5.2 REMARKS ON THE NUMERICAL SOLUTION:

The instability is predicted for large values of the parameter αRa^2. The per step truncation error involved in the numerical integration increases with αRa^2. To have a better control on the per step truncation error an appropriate step size must be chosen. This was achieved by trial. With a step size of 0.0025 two iterations can be performed in one minute. It is essential to have good approximations for the initial values for fast convergence. Tables 2, 3 and 4 give the initial values at representative points for different values of a. Lagrangian interpolation was used for determining the critical Reynolds numbers and the critical wave numbers by using the values obtained for different values of the flow parameters.

Appendix B contains a list of symbols. A brief discussion of the program is given in appendix C. A listing of the program has been given in appendix D.

5.3 CONCLUDING REMARKS:

The couple stress theory introduces a material constant ℓ which has the dimension of length. A size dependent effect is predicted by this theory which is not present in the

nonpolar case. The effects of couple stresses are quite large for small values of the parameter $a = h/\ell$. The parameter a enters the resulting equations of motion as the multiplier of the highest order space derivative. The limit $a \rightarrow \infty$, corresponds to the classical nonpolar case. The order of these equations and the number of boundary conditions is diminished in the transition to the classical theory.

From the discussions in the previous sections it follows that the effect of couple stresses is equivalent to an apparent increase in the viscosity of the fluid.

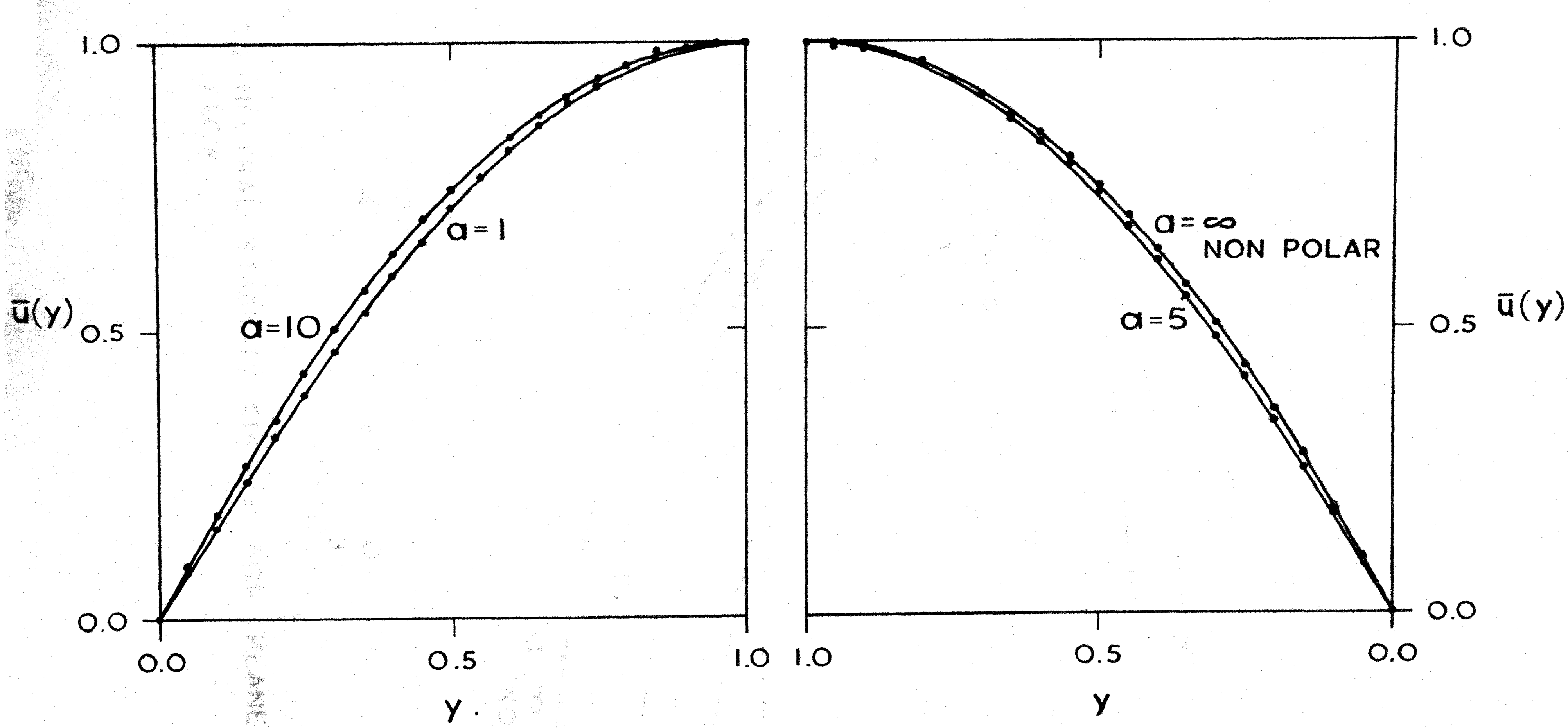

FIG.1-MEAN VELOCITY PROFILES FOR PLANE POISEUILLE FLOW FOR VARIOUS a.

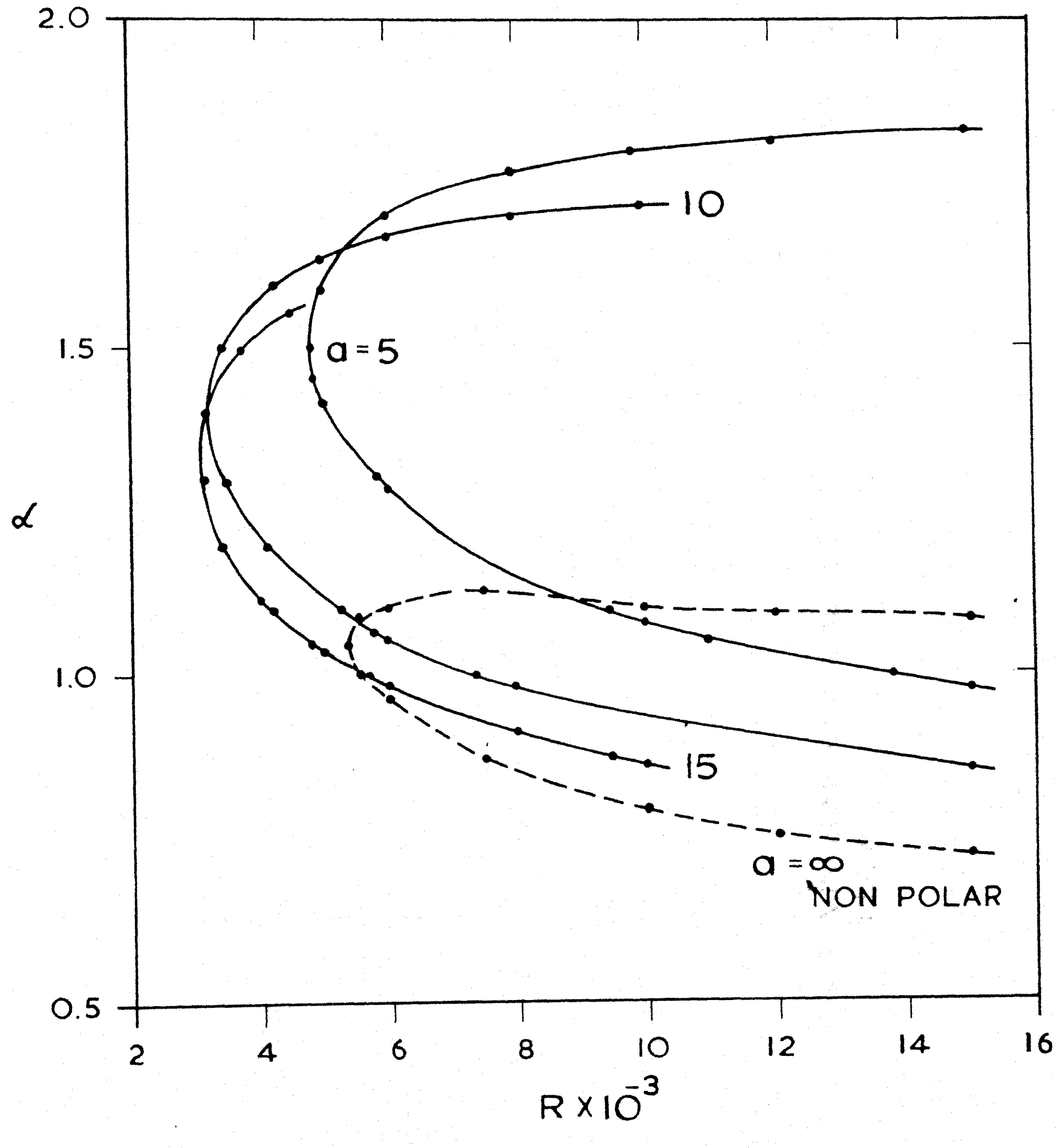

FIG. 2 - NEUTRAL STABILITY CURVES FOR PLANE POISEUILLE FLOW.

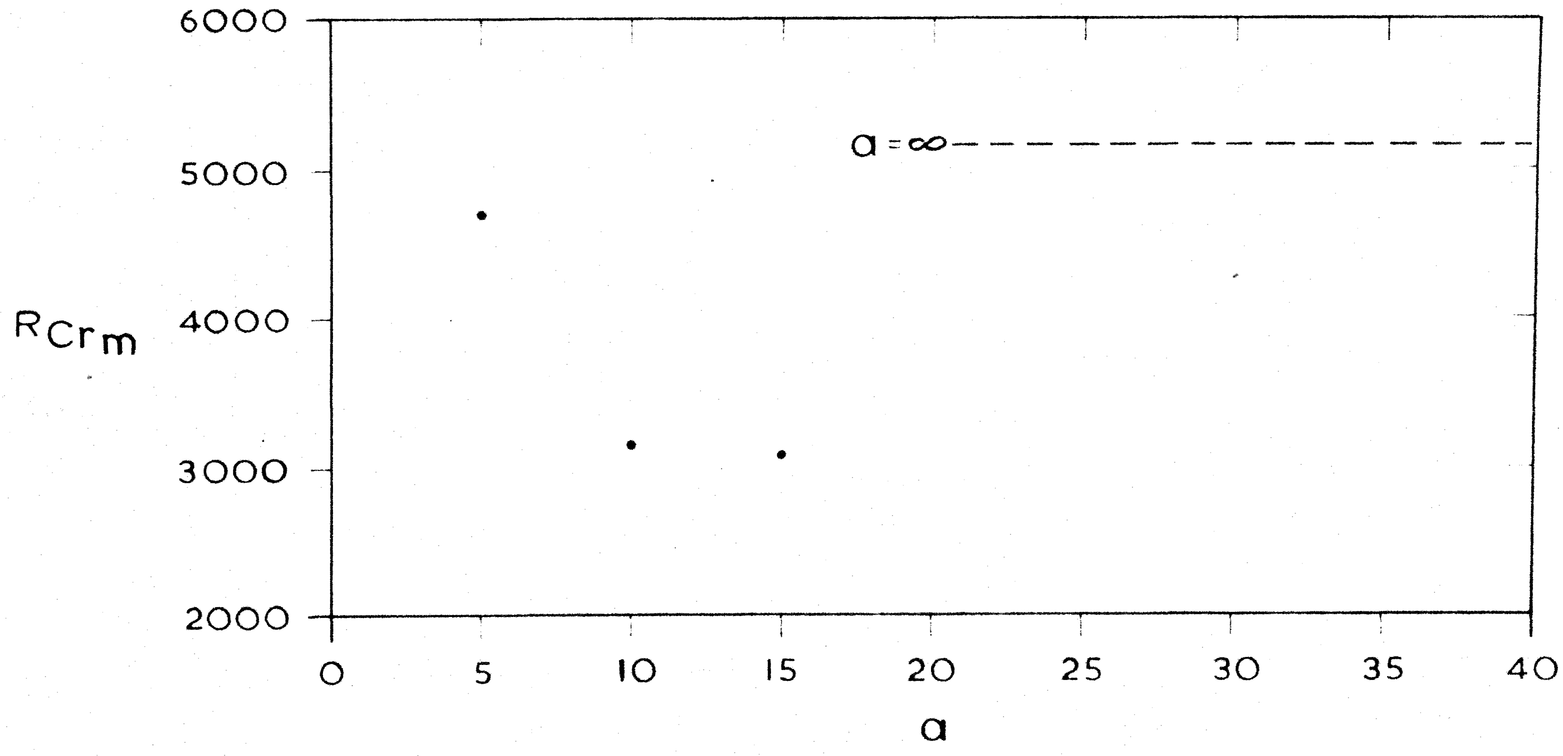

FIG. 3 - MINIMUM CRITICAL REYNOLDS NUMBER FOR PLANE POISEUILLE FLOW.

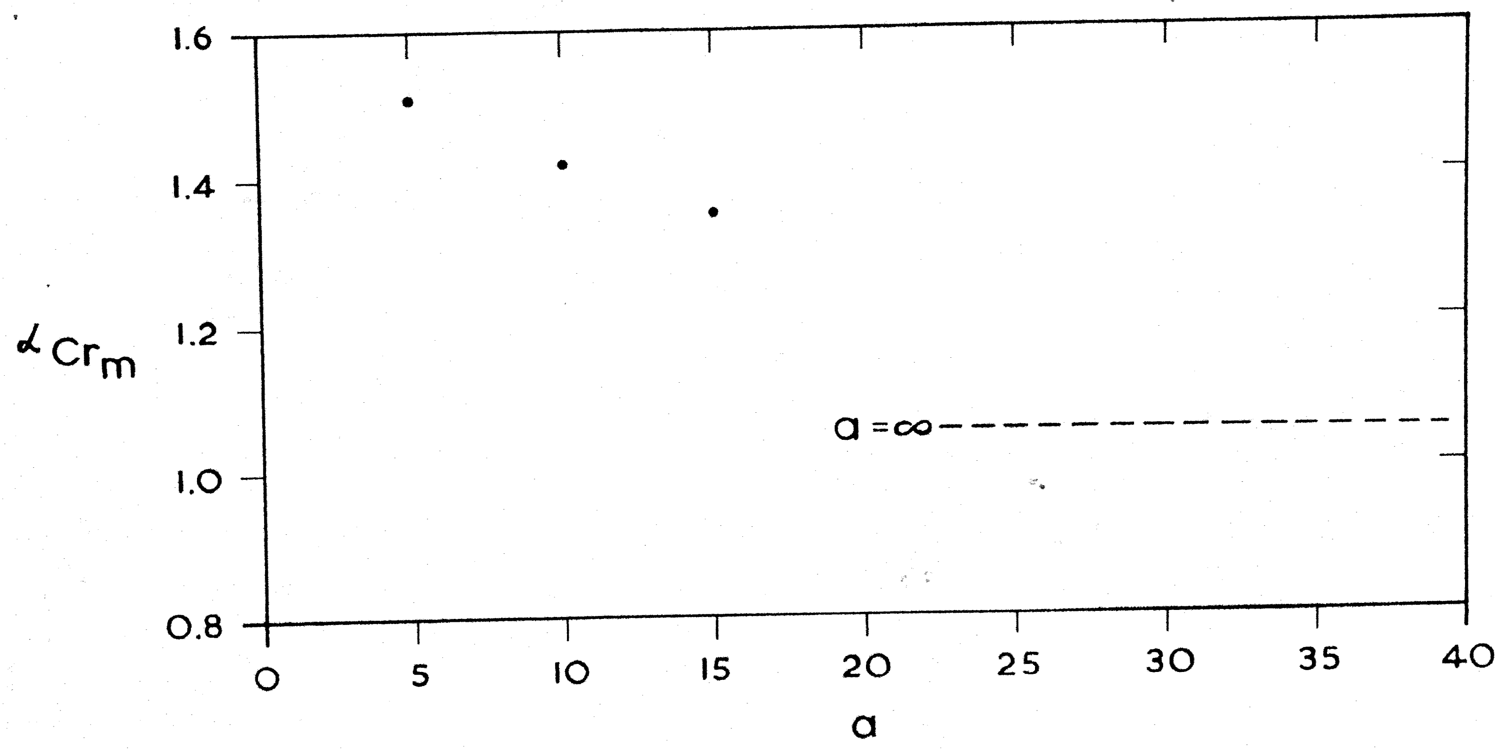

FIG. 4 - MINIMUM CRITICAL WAVE NUMBER FOR PLANE POISEUILLE FLOW.

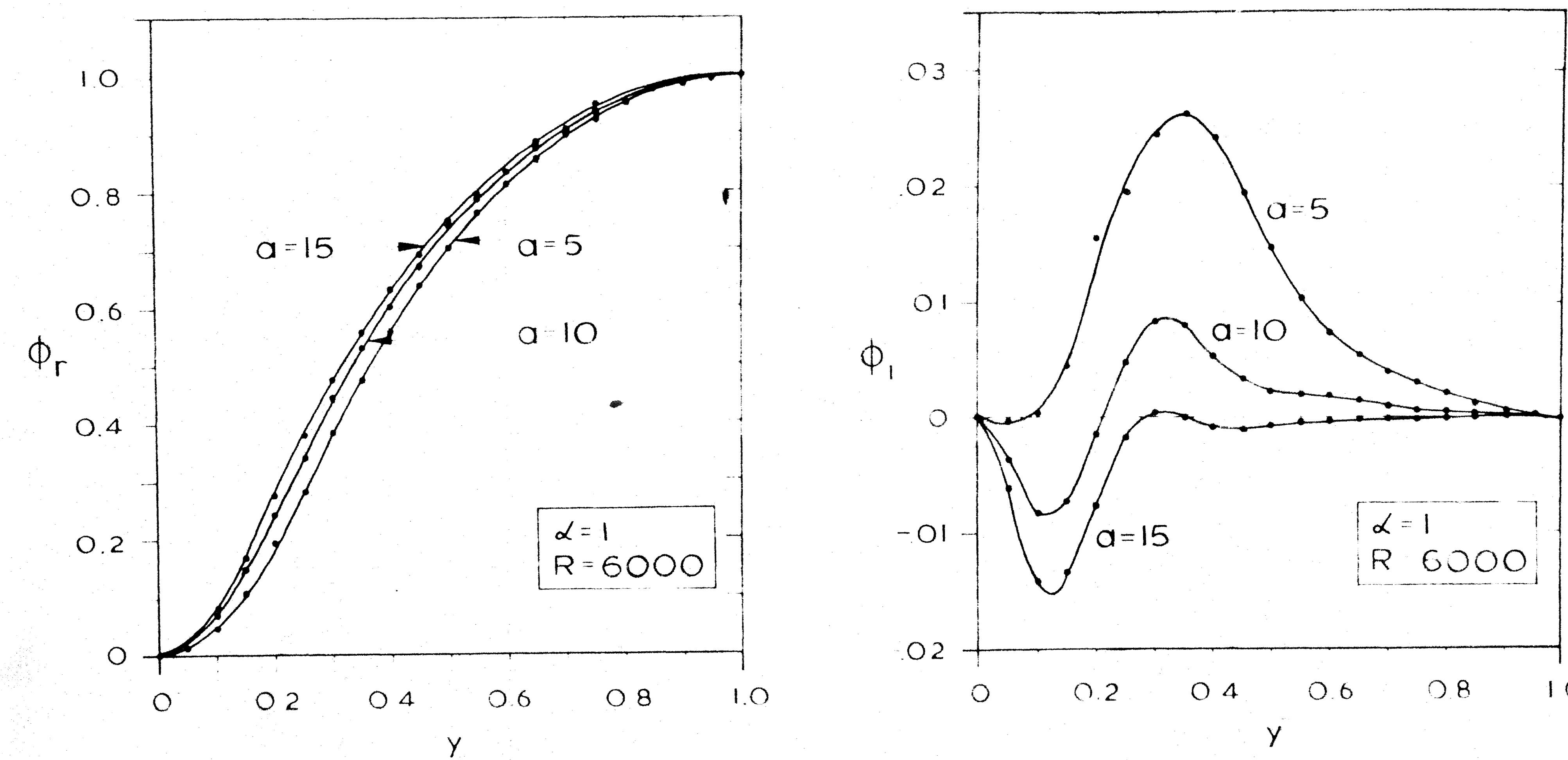

FIG 5 - MODE SHAPES OF EIGENFUNCTIONS FOR PLANE POISEUILLE FLOW FOR VARIOUS a

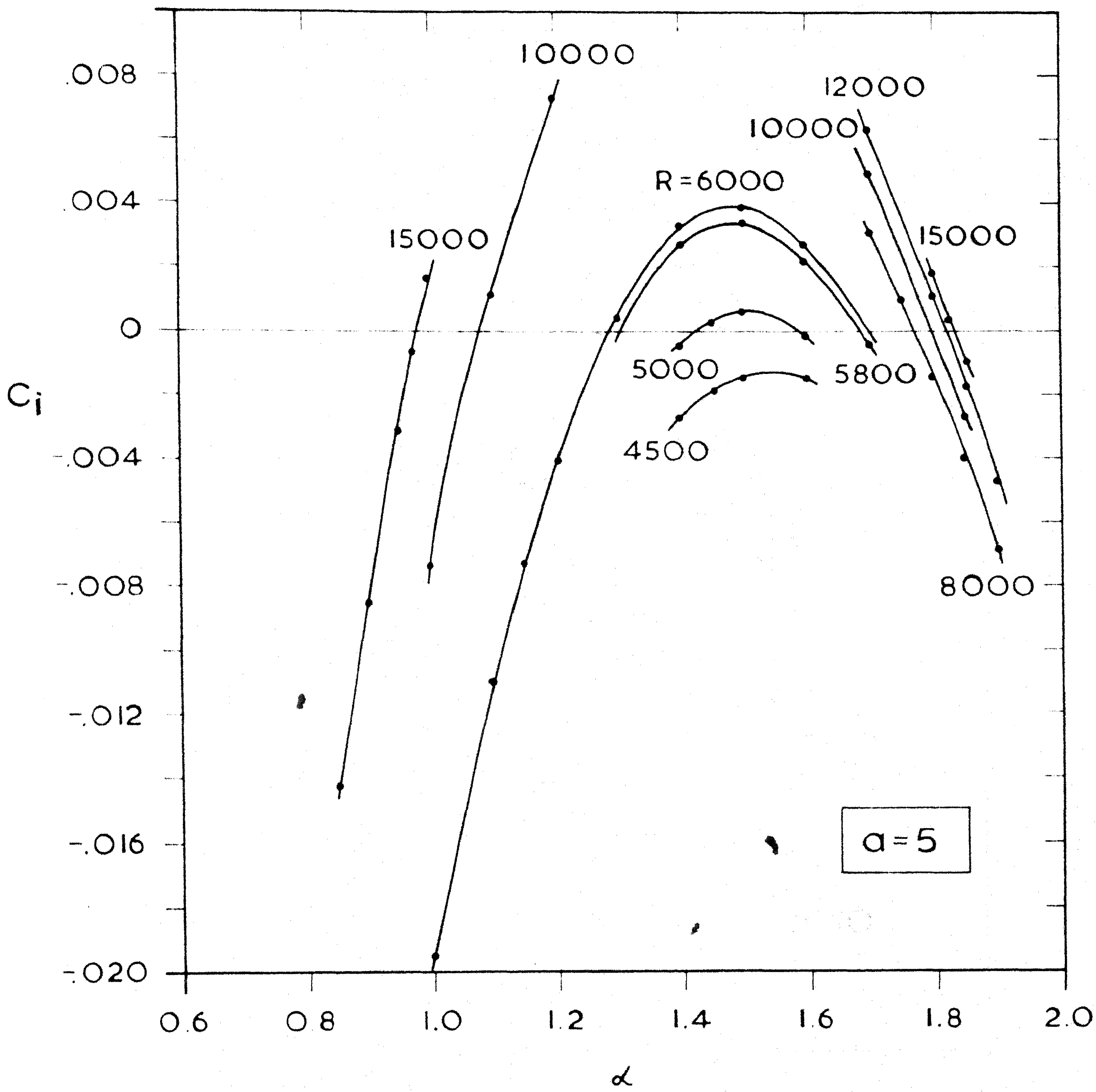

FIG. 6 - EIGENVALUES FOR PLANE POISEUILLE FLOW FOR VARIOUS R.

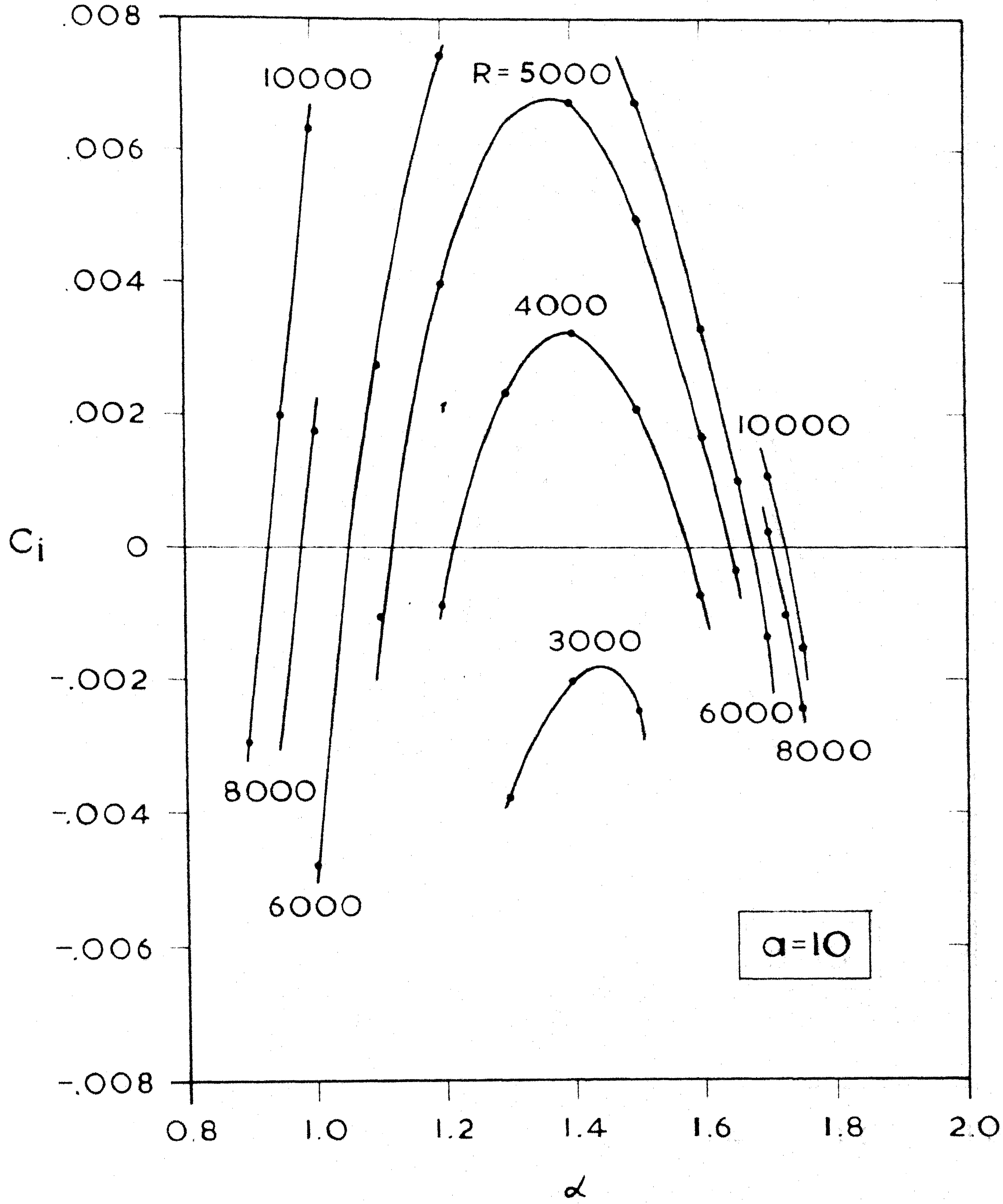

FIG. 7 - EIGENVALUES FOR PLANE POISEUILLE FLOW FOR VARIOUS R.

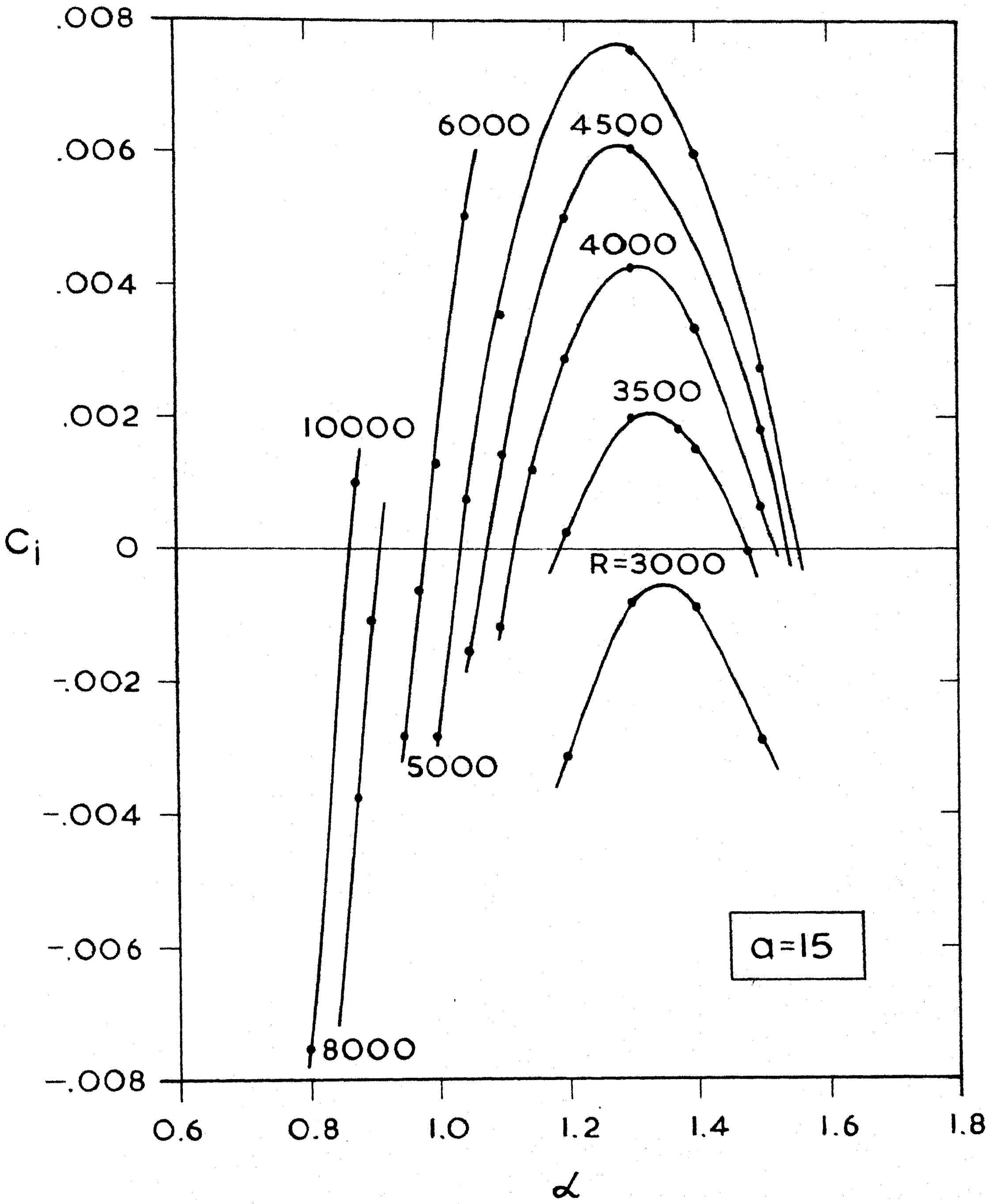

FIG. 8- EIGENVALUES FOR PLANE POISEUILLE FLOW FOR VARIOUS R.

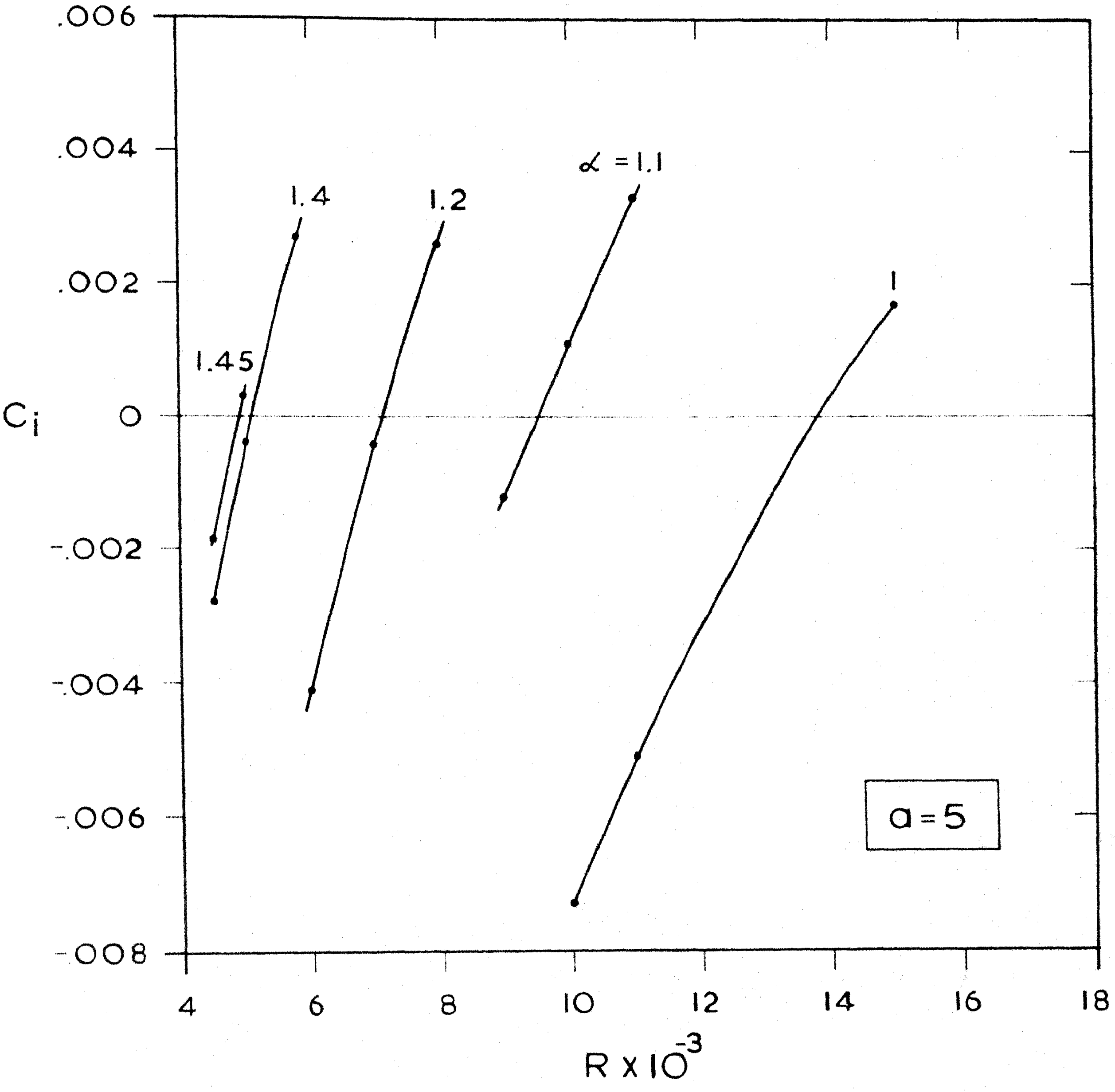

FIG. 9 - EIGENVALUES FOR PLANE POISEUILLE FLOW FOR VARIOUS α.

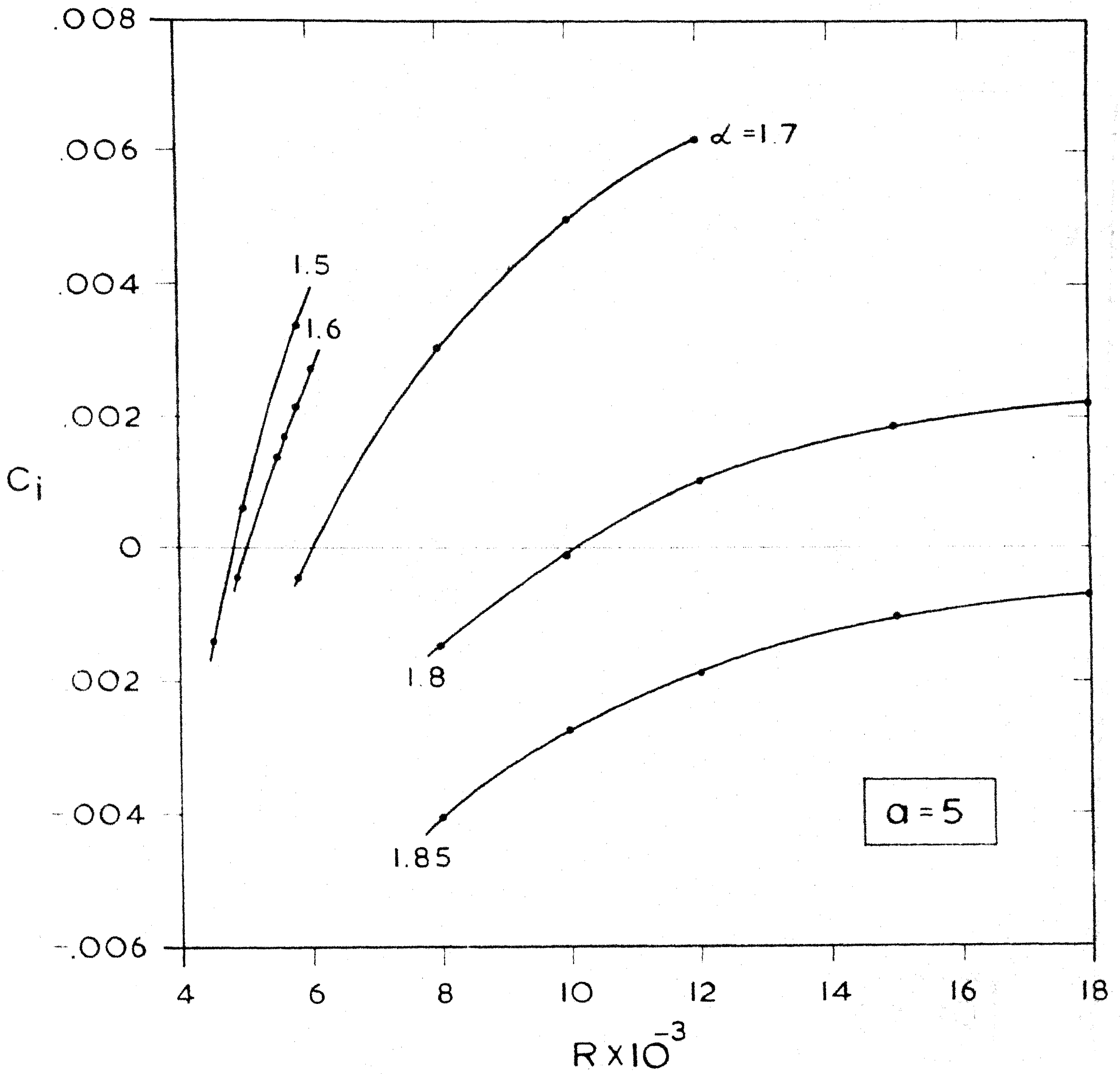

FIG.10- EIGENVALUES FOR PLANE POISEUILLE FLOW FOR VARIOUS α.

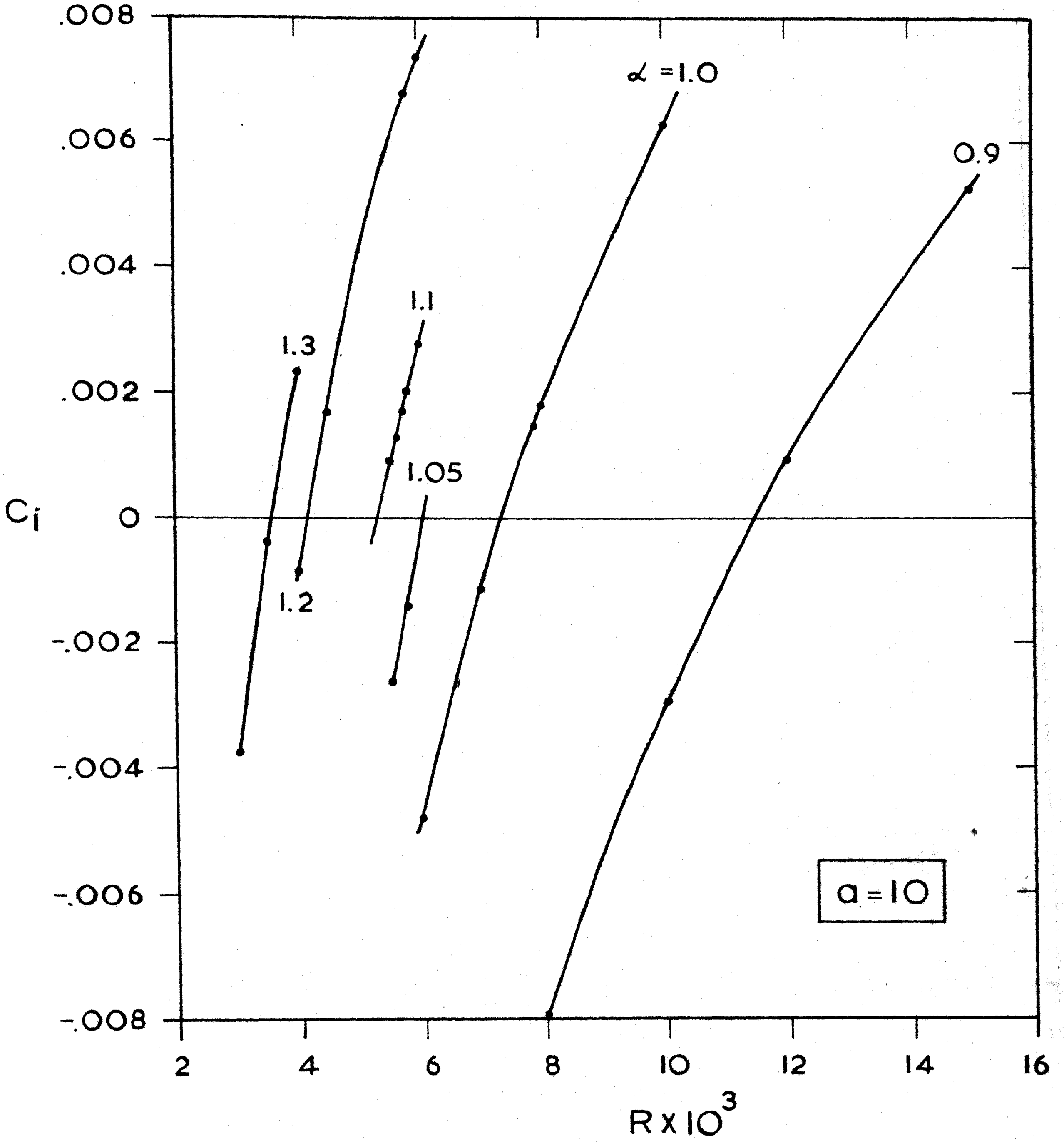

FIG. 11 - EIGENVALUES FOR PLANE POISEUILLE FLOW FOR VARIOUS α.

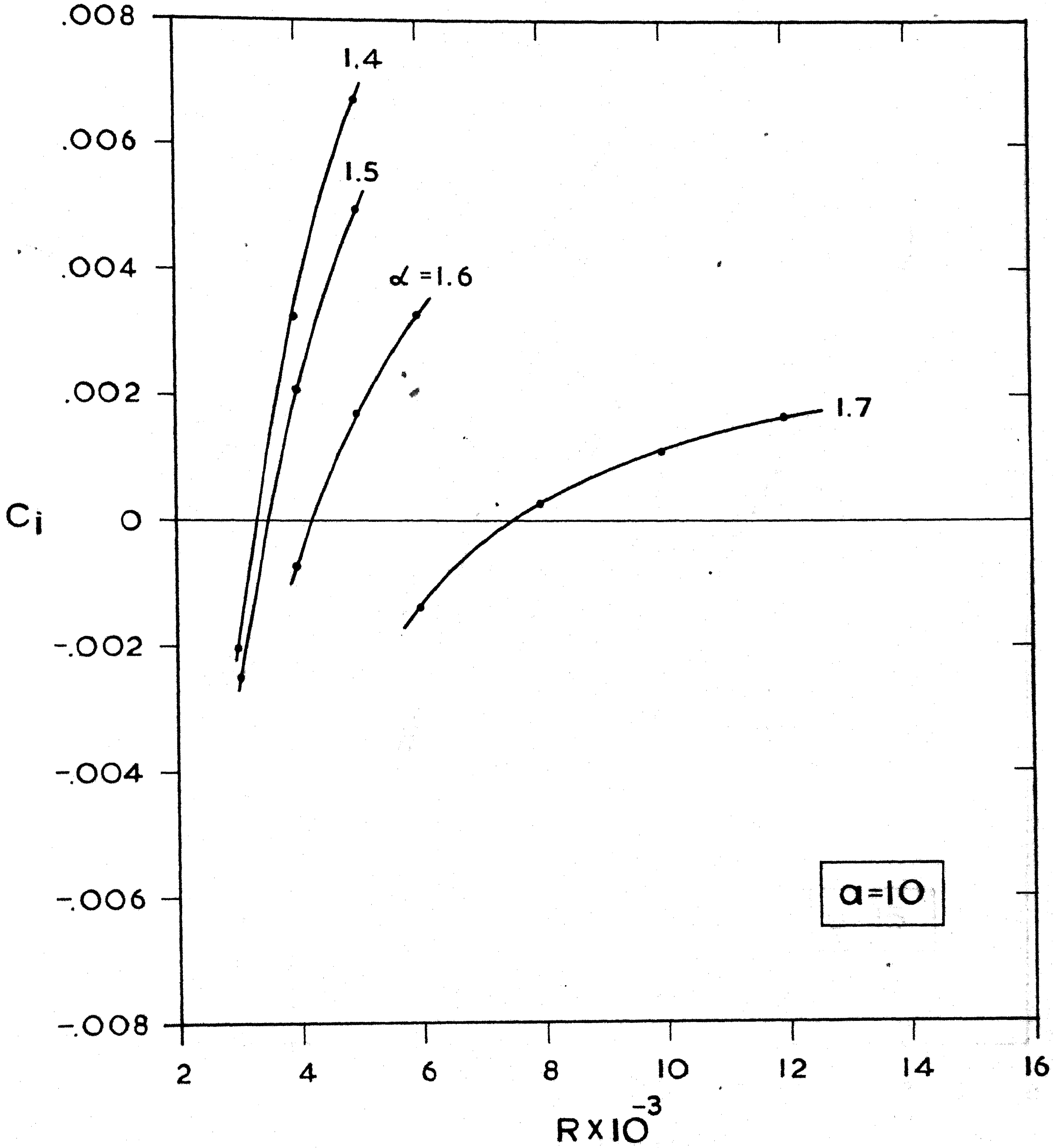

FIG.12-EIGENVALUES FOR PLANE POISEUILLE FLOW FOR VARIOUS α.

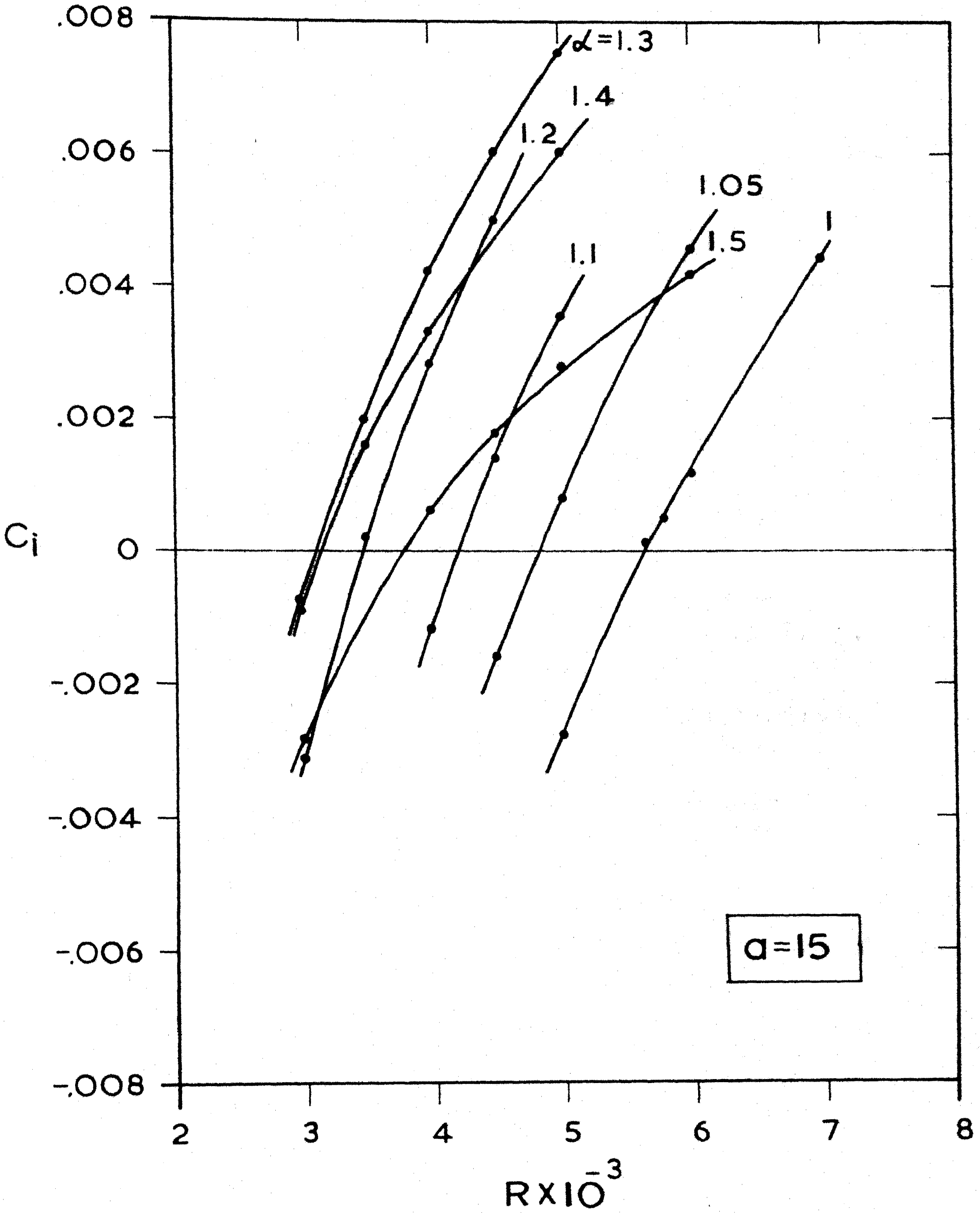

FIG.13- EIGENVALUES FOR PLANE POISEUILLE FLOW FOR VARIOUS α.

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APPENDIX A

TABLE 1

Minimum Critical Reynolds Number for Plane Poiseuille Flow

	POLAR			NONPOLAR		
a	5	10	15	Lin (1944)	Thomas (1953)	Nachtsheim (1964)
α	1.51	1.42	1.35	1.05	1.026	1.03
R	4700	3175	3100	5300	5780	5767

TABLE 2.1

a = 5

Variables	R=6000 α=1.20	R=10000 α=1.10	R=15000 α=1.0
p_r	-0.3513156047150851D+01	-0.3315444837975074D+01	-0.3177539867931746D+01
p_i	0.5700955930021749D-01	-0.9429359976180998D-02	-0.1563146624472739D-01
q_r	-0.7409850891612737D+01	-0.1915011744664627D+01	0.3944977623097087D+01
q_i	-0.6851646439581127D+01	0.4958197790872951D+01	0.2102011554048555D+01
r_r	0.1268418295669884D+02	0.1357936361023568D+02	0.1398853568146881D+02
r_i	-0.2766057183657018D+01	-0.3459398482806651D+01	-0.3470973420850016D+01
g_r	-0.8228207378070669D+03	-0.9994928781499227D+03	-0.1099765971167474D+04
g_i	0.2439139477825696D+04	0.3178844882909465D+04	0.3586132491664508D+04
t_r	0.8366908779372568D+03	-0.2112819955153514D+03	-0.4241960006966786D+03
t_i	-0.6895164645625734D+05	-0.9549999897260410D+05	-0.1145735941725946D+06
c_r	0.3908569162679212D+00	0.3565718745762839D+00	0.3256567276163306D+00
c_i	-0.4442563438942901D-02	0.1185350553124285D-02	0.1782614940407079D-02

* $D \pm XX = 10^{\pm XX}$

TABLE 2.2

$a = 5$

Variables	R=6000 α=1.70	R=12000 α=1.80	R=18000 α=1.85
p_r	-0.390184388470966 3D+01	-0.3749671084395467D+01	-0.3607344871649048D+01
p_i	0.2679411630235003D-01	-0.1630569673285222D-01	0.1321790410256050D-01
q_r	-0.1423894487536601D+02	0.1028140941574200D+02	-0.3929678929140201D+01
q_i	0.1355231721835973D+01	-0.2101077431337811D+01	-0.3939434292006357D+01
r_r	0.2033209565123428D+02	0.3165784087525341D+02	0.4136184789046936D+02
r_i	-0.6405118535647628D+01	-0.1063830448637813D+02	-0.1360648757798390D+02
g_r	-0.2143183010023889D+04	-0.4972148358125633D+04	-0.8130891322509485D+04
g_i	0.5845385685704370D+04	0.1320008971055359D+05	0.2095371804225984D+05
t_r	0.5180780381594417D+03	-0.1474449271350844D+03	0.3914394714080530D+04
t_i	-0.1738498371274684D+06	-0.4585648990088210D+06	-0.8006797614497929D+06
c_r	0.4555185857097603D+00	0.4265571408031893D+00	0.4063716900640236D+00
c_i	0.4837805871682502D-05	0.1028898935373108D-02	-0.7065397493779367D-03

TABLE 3.1

$a = 10$

Variables	R=6000 α=1.0	R=10000 α=0.90	R=15000 α=0.80
p_r	-0.301331800135429OD+01	-0.2864256027015815D+01	-0.2745671659049105D+01
p_i	0.2162536854694446D-01	0.1254854339250820D-01	0.2214284121143598D-01
q_r	0.1151097835501041D+01	-0.8987452957522096D+00	0.3305256218706285D+00
q_i	-0.1759562064766266D+01	0.1720006941368802D+00	0.2395446735897298D+00
r_r	0.1581513236839183D+02	0.1672669759765336D+02	0.1705562237986624D+02
r_i	-0.3661821561104569D+01	-0.3813560716304308D+01	-0.2917076398347382D+01
g_r	-0.1538345919406271D+04	-0.1804311711760800D+04	-0.1870945379192763D+04
g_i	0.5151965741663613D+04	0.6192091432916278D+04	0.6324220088786190D+04
t_r	0.2168970228220668D+04	0.1658121967378302D+04	0.3544231441184955D+04
t_i	-0.1888481911230943D+06	-0.2448970726461124D+06	-0.2724553426267461D+06
c_r	0.3221218675512067D+00	0.2878757240887051D+00	0.2568957409666432D+00
c_i	-0.4772394457005996D-02	-0.2960522955857574D-02	-0.6128599205973962D-02

TABLE 3.2

a = 10

Variables	R=4000 α =1.40	R=8000 α = 1.70	R=12000 α= 1.70
p_r	-0.3380066325107882D+01	-0.3305729110891307D+01	-0.3195987731333938D+01
p_i	-0.2381481446897300D-01	-0.3131452842729752D-02	-0.8070238235155407D-02
q_r	0.3377482972675873D+01	0.5697769965353168D-01	0.1735184659580536D+00
q_i	0.6199056481405932D+00	0.1812651477182007D+01	-0.1043220948607097D+01
r_r	0.2112077198153283D+02	0.4310647498709118D+02	0.5286649747653794D+02
r_i	-0.7597281896652289D+01	-0.1540024487382861D+02	-0.1938023439873159D+02
g_r	-0.2716826536161835D+04	-0.9955160711805048D+04	-0.1471596213596270D+05
g_i	0.8625063168088990D+04	0.2765695999395239D+05	0.4065494901929131D+05
t_r	-0.2376911699932346D+04	0.1262090819525333D+04	-0.5587444859674461D+04
t_i	-0.3032607461078476D+06	-0.1182147912146337D+07	-0.1882442313009439D+07
c_r	0.3957708056518083D+00	0.3830347830564935D+00	0.3614176414105971D+00
c_i	0.3255261657152383D-02	0.2797528976656477D-03	0.1691898312431020D-02

TABLE 4.1

a = 15

Variables	R = 5000 α=1.0	R = 10000 α=0.875	R =12000 α=0.80
p_r	-0.289932691564573OD+01	-0.2723807919476897D+01	-0.2655089655677172D+01
p_i	0.1860093627866191D-01	0.1604949264699023D-02	0.1802789079313392D-01
q_r	-0.7140960516632747D+01	-0.8223035938984229D+01	-0.8620714401936444D+01
q_i	-0.5301344476192719D+01	-0.6416715210748741D+01	-0.6417774119922047D+01
r_r	0.1832127695229107D+02	0.2001075187717461D+02	0.1939153621344945D+02
r_i	-0.5419488603087034D+01	-0.6075656572409681D+01	-0.4609431710144686D+01
g_r	-0.2422640323103780D+04	-0.3125384483252811D+04	-0.2857397387182632D+04
g_i	0.8782884446716851D+04	0.1163824834110156D+05	0.1058089041998638D+05
t_r	0.2432981175984435D+04	-0.1084355532872286D+04	0.3798532317984079D+04
t_i	-0.3670786828782979D+06	-0.5343035603343986D+06	-0.5058198450749658D+06
c_r	0.3126651979567664D+00	0.2687701617190127D+00	0.2500071513076952D+00
c_i	-0.2783838732363244D-02	0.1034496202308763D-02	-0.3597638721908713D-02

TABLE 4.2

a = 15

Variables	R=5000 α=1.50	R=6000 α=1.5	R=4000 α=1.5
p_r	-0.3156508431285987D+01	-0.3110155067957694D+01	-0.3215772749713820D+01
p_i	-0.7853736594746838D-02	-0.1420039528000404D-01	0.4144046947112948D-02
q_r	-0.8070841811690506D+01	-0.8610382587348978D+01	-0.8468760731763281D+01
q_i	-0.6056440421966643D+01	-0.7065276657049599D+01	-0.5719450483199648D+01
r_r	0.3560723489882289D+02	0.3896315832502179D+02	0.3189008970259040D+02
r_i	-0.1423102006743601D+02	-0.1589239260016990D+02	-0.1233461665883852D+02
g_r	-0.8220311006245342D+04	-0.9806474928379307D+04	-0.6628301976510076D+04
g_i	0.2565888355926127D+05	0.3051994537390957D+05	0.2070938374154469D+05
t_r	-0.7088589047299126D+04	-0.1389435732643185D+05	-0.3782028200693697D+03
t_i	-0.1159087957614791D+07	-0.1425627655418784D+07	-0.8989229205103871D+06
c_r	0.3688641851988189D+00	0.3595139245237229D+00	0.3804534801274720D+00
c_i	0.2758021875690775D-02	0.4162134104675487D-02	0.6225439038449522D-03

APPENDIX B

Symbols

A	$S'' - \alpha^2 S$
a	non-dimensional couple stress parameter
c	phase velocity
g	$A(0)$
h	half-width of the channel
k_{ij}	curvature-twist rate tensor
ℓ	material constant
p	$S(1)$
q	$A(1)$
r	$S(0)$
R	Reynolds number
S	disturbance vorticity amplitude
t	$A'(0)$
$\bar{U}$	velocity of basic flow
u'	disturbance velocity parallel to plates
v'	disturbance velocity normal to plates
x	distance parallel to plates
y	normal distance from lower plate
y_c	matching point
α	wave number
ω_i	vorticity vector
ϕ	stream function amplitude
Ψ	stream function

Superscripts :

*	dimensional quantities
'	denotes differentiation with respect to y
(-)	basic flow quantities
(.)	time derivative following a particle

APPENDIX C

DESCRIPTION OF THE FORTRAN PROGRAM FOR THE SOLUTION OF THE EIGENVALUE PROBLEM FOR PLANE POISEUILLE FLOW

The numerical procedure outlined previously for solving the eigenvalue problem was programmed for solution on the IBM 7044 in FORTRAN IV.

The correspondance between the FORTRAN symbols used in the program and the mathematical notation employed previously is shown in the following list :

FORTRAN symbol	Mathematical symbol	FORTRAN symbol	Mathematical symbol
Y1	ϕ_r	Y1C	ϕ_{r,c_r}
Y2	ϕ_i	Y2C	ϕ_{i,c_r}
S1	S_r	S1C	S_{r,c_r}
S2	S_i	S2C	S_{i,c_r}
A1	A_r	A1C	A_{r,c_r}
A2	A_i	A2C	A_{i,c_r}
Y1P	ϕ_{r,p_r}	Y1Q	ϕ_{r,q_r}
Y2P	ϕ_{i,p_r}	Y2Q	ϕ_{i,q_r}
S1P	S_{r,p_r}	S1Q	S_{r,q_r}
S2P	S_{i,p_r}	S2Q	S_{i,q_r}
A1P	A_{r,p_r}	A1Q	A_{r,q_r}
A2P	A_{i,p_r}	A2Q	A_{i,q_r}

FORTRAN symbol	Mathematical symbol	FORTRAN symbol	Mathematical symbol
Y1T	$\emptyset_{r,t_r}$	DA1C	A'_{r,c_r}
Y2T	$\emptyset_{i,t_r}$	DA2C	A'_{i,c_r}
S1T	S_{r,t_r}	DY1Q	$\emptyset'_{r,q_r}$
S2T	S_{i,t_r}	DY2Q	$\emptyset'_{i,q_r}$
A1T	A_{r,t_r}	DS1Q	S'_{r,q_r}
A2T	A_{i,t_r}	DS2Q	S'_{i,q_r}
DY1	$\emptyset'_r$	DA1Q	A'_{r,q_r}
DY2	$\emptyset'_i$	DA2Q	A'_{i,q_r}
DS1	S'_r	DY1T	$\emptyset'_{r,t_r}$
DS2	S'_i	DY2T	$\emptyset'_{i,t_r}$
DA1	A'_r	DS1T	S'_{r,t_r}
DA2	A'_i	DS2T	S'_{i,t_r}
DY1P	$\emptyset'_{r,p_r}$	DA1T	A'_{r,t_r}
DY2P	$\emptyset'_{i,p_r}$	DA2T	A'_{i,t_r}
DS1P	S'_{r,p_r}	C1	C_r
DS2P	S'_{i,p_r}	C2	C_i
DA1P	A'_{r,p_r}	DELP1	Δp_r
DA2P	A'_{i,p_r}	DELP2	Δp_i
DY1C	$\emptyset'_{r,c_r}$	DELQ1	Δq_r
DY2C	$\emptyset'_{i,c_r}$	DELQ2	Δq_i
DS1C	S'_{r,c_r}	DELR1	Δr_r
DS2C	S'_{i,c_r}	DELR2	Δr_i

FORTRAN symbol	Mathematical symbol	FORTRAN symbol	Mathematical symbol
DELG1	Δg_r	A1BACK	$A_r(0)$
DELG2	Δg_i	A2BACK	$A_i(0)$
DELT1	Δt_r	DA1BAK	$A_r'(0)$
DELT2	Δt_i	DA2BAK	$A_i'(0)$
DELC1	Δc_r	AL	α
DELC2	Δc_i	R	R
S1FWD	$S_r(1)$	AHL	a
S2FWD	$S_i(1)$	W	$\bar{u}$
A1FWD	$A_r(1)$	DDW	$\bar{u}''$
A2FWD	$A_i(1)$	X	y
S1BACK	$S_r(0)$	XEND	$y=1$
S2BACK	$S_i(0)$	XMATCH	y_c

The following remarks are intended to aid in a study of the program :

(1) Subroutine JKJ is used to evaluate the second derivatives. The variables Z and DDZ that appear in JKJ are dummy variables.

(2) Subroutine JAIN is used to store the matrix of coefficients that are formed from functions and partial derivatives evaluated at the matching point. The solution of the simultaneous linear equations is accomplished by calling subroutine GAUSS. Before the subroutine GAUSS is called, EE contains the coefficient matrix and VV contains the "right-hand side". After GAUSS is called, VV contains the answers.

(3) Subroutine INTEGR carries out the step-by-step integration with either the Runge-kutta method (INDEX=0) or the Milne method, which uses the Runge-kutta method to obtain starting values (INDEX=1). The program listing is given in appendix D.

APPENDIX D

```
£JOB    MEG092,TIME008,PAGES030,NAME JKJAIN
£IBJOB
£IBFTC JKDP    NOPRNT
C     ************************************************************************
C
C                    STABILITY  OF  PLANE  POISEUILLE   FLOW
C
C     ************************************************************************
      EXTERNAL JKJ
      DIMENSION T(30),DT(30),DDT(30),SS(200),VK(15),DEL(16),VVK(16)
      DIMENSION  DELT(15,150)
      DOUBLE PRECISION
     1T,DT,  S1FWD,S2FWD,A1FWD,A2FWD,S1BACK,S2BACK,A1BACK,A2BACK,DA1BAK, 402
     1DA2BAK,DELP1,DELP2,DELQ1,DELQ2,DELR1,DELR2,DELG1,DELG2,DELT1,DELT2 403
     2,DELC1,DELC2,C1,C2
      DOUBLE PRECISION Y1,Y2,S1,S2,A1,A2,Y1P,Y2P,S1P,S2P,A1P,A2P,
     1Y1C,Y2C,S1C,S2C,A1C,A2C,Y1Q,Y2Q,S1Q,S2Q,A1Q,A2Q,Y1T,Y2T,S1T,S2T,
     2A1T,A2T,DY1,DY2,DS1,DS2,DA1,DA2,DY1P,DY2P,DS1P,DS2P,DA1P,DA2P,
     3DY1C,DY2C,DS1C,DS2C,DA1C,DA2C,DY1Q,DY2Q,DS1Q,DS2Q,DA1Q,DA2Q,
     4DY1T,DY2T,DS1T,DS2T,DA1T,DA2T
      DOUBLE PRECISION  DDT,ACY1,ACY2,DEL,VK,VVK,SS,SFRESH,DELT
      COMMON T ,DT
      COMMON S1FWD,S2FWD,A1FWD,A2FWD,S1BACK,S2BACK,A1BACK,A2BACK,DA1BAK, 402
     1DA2BAK,DELP1,DELP2,DELQ1,DELQ2,DELR1,DELR2,DELG1,DELG2,DELT1,DELT2 403
     2,DELC1,DELC2
      COMMON C1,C2,AL,R,W,DDW,AA,AR,AHL                                  401
      EQUIVALENCE (Y1,T(1)),(Y2,T(2)),(S1,T(3)),(S2,T(4)),(A1,T(5)),(A2, 406
     1T(6)),(Y1P,T(7)),(Y2P,T(8)),(S1P,T(9)),(S2P,T(10)),(A1P,T(11)),(A2 407
     2P,T(12)),(Y1C,T(13)),(Y2C,T(14)),(S1C,T(15)),(S2C,T(16)),(A1C,T(17 408
     3)),(A2C,T(18)),(Y1Q,T(19)),(Y2Q,T(20)),(S1Q,T(21)),(S2Q,T(22)),(A1
     4Q,T(23)),(A2Q,T(24)),(Y1T,T(25)),(Y2T,T(26)),(S1T,T(27)),(S2T,T(28)),
     5)),(A1T,T(29)),(A2T,T(30))
      EQUIVALENCE (DY1,DT(1)),(DY2,DT(2)),(DS1,DT(3)),(DS2,DT(4)),
     1(DA1,DT(5)),(DA2,DT(6)),(DY1P,DT(7)),(DY2P,DT(8)),(DS1P,DT(9)),
     2(DS2P,DT(10)),(DA1P,DT(11)),(DA2P,DT(12)),(DY1C,DT(13)),(DY2C,DT(14)),
     34)),(DS1C,DT(15)),(DS2C,DT(16)),(DA1C,DT(17)),(DA2C,DT(18)),
     4(DY1Q,DT(19)),(DY2Q,DT(20)),(DS1Q,DT(21)),(DS2Q,DT(22)),
     5(DA1Q,DT(23)),(DA2Q,DT(24)),(DY1T,DT(25)),(DY2T,DT(26)),(DS1T,DT(27)),(DS
     67)),(DS2T,DT(28)),(DA1T,DT(29)),(DA2T,DT(30))
C     ************************************************************************
      READ 70,INDEX,NFWD,NBACK,NR,NAL,NAHL,ITERAT
 70   FORMAT( 7I5)
      READ 71,H,DELX,XEND,XMATCH                                         420
   71  FORMAT(4F6.4)                                                     421
      READ  72,ACY1,FR,FTN,IPRINT
 72   FORMAT(3F8.4, I5)
      READ 73,HR,HAHL,HAL
 73   FORMAT(3F14.6)
      READ 74,AL,AHL,R
 74   FORMAT(F12.8,F12.8,F12.4)                                          425
```

```
      READ75,S1FWD,S2FWD,A1FWD,A2FWD,S1BACK,S2BACK,A1BACK,A2BACK,DA1BAK,
     1DA2BAK,C1,C2
 75   FORMAT(3D24.16)
      PRINT 130
      PRINT 76,INDEX,NFWD,NBACK,NR,NAL,NAHL,ITERAT
 76   FORMAT(//2X,*INDEX=*,I5,2X,*NFWD=*,I5,2X,*NBACK=*,I5,2X,*NR=*,I5,
     1 2X,*NAL=*,I5,2X,*NAHL=*,I5,2X,*ITERAT=*,I5)
      PRINT 77,H,DELX,XEND,XMATCH
 77   FORMAT(//2X,3H H=F12.8,2X,5HDELX=F12.8,2X,5HXEND=F12.8,2X,7HXMATCH
     1=F12.8)
      PRINT 78,ACY1,FR,FTN,IPRINT
 78   FORMAT(//2X,*ACY1=*,F12.8,2X,*FR  =*,F12.8,2X,*FTN=*,F12.8,
     1 2X,*IPRINT=*,I5)
      PRINT 79,HR,HAHL,HAL
 79   FORMAT(//2X,3HHR=F8.2,5X,5HHAHL=F12.8,5X,4HHAL=F12.4)
      PRINT 130
C      ***********************************************************************
      FTC=FTN/5.0
      FRC=FR
      FTNS=FTN
      IPRNT=IPRINT
      IRT=ITERAT
      CALL FLUN(5000)
      CALL FLOV(5000)
      AST=0.
      BST=0.
      CST=0.
      DO 100 KAHL=1,NAHL
      AHL=AHL+BST
      DO 100 KAL=1,NAL
      AL=AL+CST
      DO 100 KR=1,NR
      R=R+AST
      ITERAT=IRT
      PRINT 80,AL,AHL,R
 80   FORMAT(/2X,3HAL=F12.8,2X,4HAHL=F12.8,2X,2HR=F12.4/)
      PRINT 81,S1FWD,S2FWD,A1FWD,A2FWD,S1BACK,S2BACK,A1BACK,A2BACK,DA1BAK
     1K,DA2BAK,C1,C2
 81   FORMAT(2X,3D24.16)
C     -----------------------------------------------------------------------
      AA=AL**2
      AR=AL*R
      JK=1
      IFINAL=2
C     -----------------------------------------------------------------------
 110  I=1
      J=1
      JK=JK+1
  49  CONTINUE
      Y1P=0.
      Y2P=0.
      DY1P=0.
      DY2P=0.
      S1P=1.
      S2P=0.
      DS1P=0.
```

```
      DS2P=0.                                                          466
      A1P=0.                                                           467
      A2P=0.                                                           468
      DA1P=0.                                                          469
      DA2P=0.                                                          470
      Y1C=0.                                                           471
      Y2C=0.                                                           472
      DY1C=0.                                                          477
      DY2C=0.                                                          478
      S1C=0.                                                           479
      S2C=0.                                                           480
      DS1C=0.                                                          481
      DS2C=0.                                                          482
      A1C=0.                                                           483
      A2C=0.                                                           484
      DA1C=0.                                                          485
      DA2C=0.                                                          486
      Y1Q=0.
      Y2Q=0.
      DY1Q=0.
      DY2Q=0.
      S1Q=0.
      S2Q=0.
      DS1Q=0.
      DS2Q=0.
      A1Q=1.
      A2Q=0.
      DA1Q=0.
      DA2Q=0.
      GO TO (50,51),J                                                  487
   50 J=2                                                              488
C     FORWARD SOLUTION                                                 489
      N=NFWD                                                           490
      M=1                                                              491
      X=XEND                                                           525
      H=-ABS(H)                                                        526
      XPRINT=XEND
      DELX=-ABS(DELX)                                                  528
      Y1=1.0                                                           497
      Y2=0.                                                            498
      DY1=0.                                                           499
      DY2=0.                                                           500
      S1=S1FWD                                                         501
      S2=S2FWD                                                         502
      DS1=0.                                                           503
      DS2=0.                                                           504
      A1=A1FWD                                                         505
      A2=A2FWD                                                         506
      DA1=0.                                                           507
      DA2=0.                                                           508
      GO TO 60                                                         509
   51  J=1                                                             510
C     BACKWARD SOLUTION                                                511
      N=NBACK                                                          512
      M=2                                                              513
      Y1T=0.
```

```
      PRINT 130
 130  FORMAT(/2X,120(1H-))
C     ***********************************************************************
      DEL(1)=DELP1
      DEL(2)=DELP2
      DEL(3)=DELQ1
      DEL(4)=DELQ2
      DEL(5)=DELR1
      DEL(6)=DELR2
      DEL(7)=DELG1
      DEL(8)=DELG2
      DEL(9)=DELT1
      DEL(10)=DELT2
      DEL(11)=DELC1
      DEL(12)=DELC2
      DO 1005 II=1,12
 1005 DELT(II,JK)=DEL(II)
      IF(IFINAL.EQ.1)GO TO 105
      SS(JK)=Y1
      IF(JK.EQ.2)GO TO 475
      IF(SS(JK)-SS(JK-1))475,375,375
 475  VK(1)=S1FWD
      VK(2)=S2FWD
      VK(3)=A1FWD
      VK(4)=A2FWD
      VK(5)=S1BACK
      VK(6)=S2BACK
      VK(7)=A1BACK
      VK(8)=A2BACK
      VK(9)=DA1BAK
      VK(10)=DA2BAK
      VK(11)=C1
      VK(12)=C2
      DO 477 I=1,12
 477  VVK(I)=VK(I)
      IPRINT=IPRNT
      JN=1
      ICOUNT=0
      FR=FRC
      FTN=FTNS
      IF(Y1-ACY1)105,105,275
 375  SS(JK)=SS(JK-1)
      IPRINT=IPRINT+1
      JN=2
      ICOUNT=ICOUNT+1
      FTN=FTC
      IF(ICOUNT.GE.2)FTN=FR
      IF(ICOUNT.GT.2)FR=FR/2.
      DO 376 I=1,12
      DELT(I,JK)=DELT(I,JK-1)
      DEL(I)=DELT(I,JK)
 376  VK(I)=VVK(I)+FTN*DEL(I)
      GO TO 405
 275  CONTINUE
      DO 276 I=1,12
 276  VK(I)=VVK(I)+FTN*DEL(I)
```

```
405   CONTINUE
      S1FWD=VK(1)
      S2FWD=VK(2)
      A1FWD=VK(3)
      A2FWD=VK(4)
      S1BACK=VK(5)
      S2BACK=VK(6)
      A1BACK=VK(7)
      A2BACK=VK(8)
      DA1BAK=VK(9)
      DA2BAK=VK(10)
      C1=VK(11)
      C2=VK(12)
      PRINT 125,ITERAT,FTN,JN,FR
 125  FORMAT(10X,*ITERAT =*,I5,20X,*FTN =*,F8.4,20X,*JN =*,I5,5X,*FR=*,F8.4
     1 8.4)
      PRINT 81,S1FWD,S2FWD,A1FWD,A2FWD,S1BACK,S2BACK,A1BACK,A2BACK,DA1BAK448
     1K,DA2BAK,C1,C2                                                    449
      ITERAT=ITERAT+1
      IF(ITERAT-150)55,55,105
 55   IF(Y1-ACY1)105,105,110
 105  AST=HR                                                            581
      BST=HAHL
      CST=HAL
      PRINT 140
 140  FORMAT(/2X,120(1H*)//20X,15HFINAL   RESULTS//2X,120(1H*))
      PRINT 80,AL,AHL,R
      PRINT 81,S1FWD,S2FWD,A1FWD,A2FWD,S1BACK,S2BACK,A1BACK,A2BACK,DA1BAK448
     1K,DA2BAK,C1,C2
      PRINT 54,Y1,Y2
      PRINT 130
      IFINAL=IFINAL-1
      IF(IFINAL.EQ.1)GO TO 110
 100   CONTINUE                                                         583
      STOP
      END                                                               585
£IBFTC INTDP   NOPRNT
      SUBROUTINE INTEGR(M,N,H,X,ISET,Y,DY,DDY,INDEX,JAI)                  2
      DIMENSION Y(30),DY(30),DDY(30),YLLL(30),YLL(30),YL(30),YR(30),
     1DYR(30),DYL(30),DDYLL(30),DDYL(30),DDYR(30),RK2(30),RK3(30),
     2P(30),T(30),DT(30)
      DOUBLE PRECISION
     1T,DT,  S1FWD,S2FWD,A1FWD,A2FWD,S1BACK,S2BACK,A1BACK,A2BACK,DA1BAK, 402
     1DA2BAK,DELP1,DELP2,DELQ1,DELQ2,DELR1,DELR2,DELG1,DELG2,DELT1,DELT2 403
     2,DELC1,DELC2,C1,C2
      DOUBLE PRECISION E,YLLL,YLL,YL,Y,DYL,DY,DDYLL,DDYL,DDY,YR,DYR,
     1 DDYR,RK2,RK3,P
      COMMON T ,DT
      COMMON S1FWD,S2FWD,A1FWD,A2FWD,S1BACK,S2BACK,A1BACK,A2BACK,DA1BAK, 402
     1DA2BAK,DELP1,DELP2,DELQ1,DELQ2,DELR1,DELR2,DELG1,DELG2,DELT1,DELT2 403
     2,DELC1,DELC2
      COMMON C1,C2,AL,R,W,DDW,AA,AR,AHL                                  2A
      E=H                                                                 5
      IF(ISET)30,30,31                                                    6
 30   IF(INDEX)32,32,34
  32  K=1                                                                 8
```

```
33    CALL JAI(M,X,Y,DDY)
      GO TO 145
34    K=2
      GO TO 33
31    GO TO (35,36,36,36,37),K
35    DO 38 I=1,N
38    P(I)=Y(I)+(H/2.)*DY(I)+(H*H/8.)*DDY(I)
      CALL JAI(M,X+H/2.,P,RK2)
      DO 39 I=1,N
39    P(I)=Y(I)+H*DY(I)+(H*H/2.)*RK2(I)
      CALL JAI(M,X+H,P,RK3)
      DO 41 I=1,N
      YR(I)=Y(I)+H*(DY(I)+(E/6.)*(DDY(I)+2.*RK2(I)))
 41   DYR(I)=DY(I)+(E/6.)*(DDY(I)+4.*RK2(I)+RK3(I))
      CALL JAI(M,X+H,YR,DDYR)
 48   CONTINUE
      DO 42 I=1,N
      GO TO (60,60,61,62,63),K
 60   Y(I)=YR(I)
      DY(I)=DYR(I)
      DDY(I)=DDYR(I)
      GO TO 42
 61   YLL(I)=0.
      YL(I)=0.
      DDYL(I)=0.
      GO TO 63
 62   YLL(I)=0.
      GO TO 63
 63   YLLL(I)=YLL(I)
      YLL(I)=YL(I)
      YL(I)=Y(I)
      Y(I)=YR(I)
      DYL(I)=DY(I)
      DY(I)=DYR(I)
      DDYLL(I)=DDYL(I)
      DDYL(I)=DDY(I)
      DDY(I)=DDYR(I)
  42  CONTINUE
 145  RETURN
 36   K=K+1
      GO TO 35
 37   DO 43 I=1,N
 43   P(I)=Y(I)+YLL(I)-YLLL(I)+(H*H/4.)*(5.*DDY(I)+2.*DDYL(I)-5.*DDYLL(I))
     1))
      CALL JAI(M,X+H,P,DDYR)
      DO 44 I=1,N
 44   YR(I)=2.*Y(I)-YL(I)+(E*E/12.)*(DDYR(I)+10.*DDY(I)+DDYL(I))
      CALL JAI(M,X+H,YR,DDYR)
      DO 45 I=1,N
 45   DYR(I)=DYL(I)+(E/3.)*(DDYR(I)+4.*DDY(I)+DDYL(I))
      GO TO 48
      END
£IBFTC JKJDP    NOPRNT
      SUBROUTINE  JKJ(M,X,Z,DDZ)
      DIMENSION T(30),DT(30),Z(30),DDZ(30)
      DOUBLE PRECISION
```

```
   1T,DT,  S1FWD,S2FWD,A1FWD,A2FWD,S1BACK,S2BACK,A1BACK,A2BACK,DA1BAK, 402
   1DA2BAK,DELP1,DELP2,DELQ1,DELQ2,DELR1,DELR2,DELG1,DELG2,DELT1,DELT2 403
   2,DELC1,DELC2,C1,C2
    DOUBLE PRECISION Z,DDZ
    COMMON T ,DT
    COMMON S1FWD,S2FWD,A1FWD,A2FWD,S1BACK,S2BACK,A1BACK,A2BACK,DA1BAK, 402
   1DA2BAK,DELP1,DELP2,DELQ1,DELQ2,DELR1,DELR2,DELG1,DELG2,DELT1,DELT2 403
   2,DELC1,DELC2
    COMMON C1,C2,AL,R,W,DDW,AA,AR,AHL                                  102
    AHLX=AHL*X                                                         104
    AAHL=AHL*AHL                                                       105
    BP=AL*R*AAHL                                                       106
    BNUM=1.-COSH(AHLX)+TANH(AHL)*SINH(AHLX)
    BDEN=1.-2./AAHL*(1.-1./COSH(AHL))
    BRATIO=BNUM/BDEN                                                   113
    W=(2.*X-X**2-2./AAHL*BNUM)/BDEN                                    114
    DDW=-2.*BRATIO                                                     115
    SUM=AAHL+AA                                                        116
    DDZ(1)=AA*Z(1)+Z(3)                                                117
    DDZ(2)=AA*Z(2)+Z(4)                                                118
    DDZ(3)=AA*Z(3)+Z(5)                                                119
    DDZ(4)=AA*Z(4)+Z(6)                                                120
    DDZ(5)=SUM*Z(5)+BP*(-Z(3)*C2+Z(4)*(W-C1)-DDW*Z(2))                 121
    DDZ(6)=SUM*Z(6)+BP*(-Z(4)*C2-Z(3)*(W-C1)+DDW*Z(1))                 122
    DDZ(7)=AA*Z(7)+Z(9)                                                123
    DDZ(8)=AA*Z(8)+Z(10)                                               124
    DDZ(9)=AA*Z(9)+Z(11)                                               125
    DDZ(10)=AA*Z(10)+Z(12)                                             126
    DDZ(11)=SUM*Z(11)+BP*(-Z(9)*C2+Z(10)*(W-C1)-DDW*Z(8))              127
    DDZ(12)=SUM*Z(12)+BP*(-Z(10)*C2-Z(9)*(W-C1)+DDW*Z(7))              128
    DDZ(13)=AA*Z(13)+Z(15)                                             129
    DDZ(14)=AA*Z(14)+Z(16)                                             130
    DDZ(15)=AA*Z(15)+Z(17)                                             131
    DDZ(16)=AA*Z(16)+Z(18)                                             132
    DDZ(17)=SUM*Z(17)+BP*(-Z(15)*C2+Z(16)*(W-C1)-DDW*Z(14)-Z(4))       133
    DDZ(18)=SUM*Z(18)+BP*(-Z(16)*C2-Z(15)*(W-C1)+DDW*Z(13)+Z(3))       134
    DDZ(19)=AA*Z(19)+Z(21)                                             137
    DDZ(20)=AA*Z(20)+Z(22)                                             138
    DDZ(21)=AA*Z(21)+Z(23)
    DDZ(22)=AA*Z(22)+Z(24)                                             1
    DDZ(23)=SUM*Z(23)+BP*(-Z(21)*C2+Z(22)*(W-C1)-DDW*Z(20))            141
    DDZ(24)=SUM*Z(24)+BP*(-Z(22)*C2-Z(21)*(W-C1)+DDW*Z(19))            142
    GO TO (63,64),M
 64 CONTINUE
    DDZ(25)=AA*Z(25)+Z(27)
    DDZ(26)=AA*Z(26)+Z(28)
    DDZ(27)=AA*Z(27)+Z(29)
    DDZ(28)=AA*Z(28)+Z(30)
    DDZ(29)=SUM*Z(29)+BP*(-Z(27)*C2+Z(28)*(W-C1)-DDW*Z(26))
    DDZ(30)=SUM*Z(30)+BP*(-Z(28)*C2-Z(27)*(W-C1)+DDW*Z(25))
 63 RETURN                                                             143
    END                                                                144
£IBFTC JAINDP   NOPRNT
    SUBROUTINE JAIN(I,IPRINT)                                          200
    DIMENSION T(30),DT(30),EE(12,12),VV(12,1)
    DOUBLE PRECISION
```

```
     1T,DT,  S1FWD,S2FWD,A1FWD,A2FWD,S1BACK,S2BACK,A1BACK,A2BACK,DA1BAK, 402
     1DA2BAK,DELP1,DELP2,DELQ1,DELQ2,DELR1,DELR2,DELG1,DELG2,DELT1,DELT2 403
     2,DELC1,DELC2,C1,C2
      DOUBLE PRECISION Y1,Y2,S1,S2,A1,A2,Y1P,Y2P,S1P,S2P,A1P,A2P,
     1Y1C,Y2C,S1C,S2C,A1C,A2C,Y1Q,Y2Q,S1Q,S2Q,A1Q,A2Q,Y1T,Y2T,S1T,S2T,
     2A1T,A2T,DY1,DY2,DS1,DS2,DA1,DA2,DY1P,DY2P,DS1P,DS2P,DA1P,DA2P,
     3DY1C,DY2C,DS1C,DS2C,DA1C,DA2C,DY1Q,DY2Q,DS1Q,DS2Q,DA1Q,DA2Q,
     4DY1T,DY2T,DS1T,DS2T,DA1T,DA2T
      DOUBLE PRECISION EE,VV
      COMMON T ,DT
      COMMON S1FWD,S2FWD,A1FWD,A2FWD,S1BACK,S2BACK,A1BACK,A2BACK,DA1BAK, 402
     1DA2BAK,DELP1,DELP2,DELQ1,DELQ2,DELR1,DELR2,DELG1,DELG2,DELT1,DELT2 403
     2,DELC1,DELC2
      COMMON C1,C2,AL,R,W,DDW,AA,AR,AHL                                  202
      EQUIVALENCE (Y1,T(1)),(Y2,T(2)),(S1,T(3)),(S2,T(4)),(A1,T(5)),(A2, 206
     1T(6)),(Y1P,T(7)),(Y2P,T(8)),(S1P,T(9)),(S2P,T(10)),(A1P,T(11)),(A2 207
     2P,T(12)),(Y1C,T(13)),(Y2C,T(14)),(S1C,T(15)),(S2C,T(16)),(A1C,T(17 208
     3)),(A2C,T(18)),(Y1Q,T(19)),(Y2Q,T(20)),(S1Q,T(21)),(S2Q,T(22)),(A1
     4Q,T(23)),(A2Q,T(24)),(Y1T,T(25)),(Y2T,T(26)),(S1T,T(27)),(S2T,T(28)),
     5)),(A1T,T(29)),(A2T,T(30))
      EQUIVALENCE (DY1,DT(1)),(DY2,DT(2)),(DS1,DT(3)),(DS2,DT(4)),
     1(DA1,DT(5)),(DA2,DT(6)),(DY1P,DT(7)),(DY2P,DT(8)),(DS1P,DT(9)),
     2(DS2P,DT(10)),(DA1P,DT(11)),(DA2P,DT(12)),(DY1C,DT(13)),(DY2C,DT(14)),
     34)),(DS1C,DT(15)),(DS2C,DT(16)),(DA1C,DT(17)),(DA2C,DT(18)),
     4(DY1Q,DT(19)),(DY2Q,DT(20)),(DS1Q,DT(21)),(DS2Q,DT(22)),
     5(DA1Q,DT(23)),(DA2Q,DT(24)),(DY1T,DT(25)),(DY2T,DT(26)),(DS1T,DT(27)),(DS
     67)),(DS2T,DT(28)),(DA1T,DT(29)),(DA2T,DT(30))
      IF(I)52,52,54                                                      216
C     FORWARD 54                                                         217
 54   VV(1,1)=Y1                                                         218
      VV(2,1)=Y2                                                         219
      VV(3,1)=DY1                                                        220
      VV(4,1)=DY2                                                        221
      VV(5,1)=S1                                                         222
      VV(6,1)=S2                                                         223
      VV(7,1)=DS1                                                        224
      VV(8,1)=DS2                                                        225
      VV(9,1)=A1                                                         226
      VV(10,1)=A2                                                        227
      VV(11,1)=DA1                                                       228
      VV(12,1)=DA2                                                       229
      EE(1,1)=Y1P                                                        230
      EE(2,1)=Y2P                                                        231
      EE(3,1)=DY1P                                                       232
      EE(4,1)=DY2P                                                       233
      EE(5,1)=S1P                                                        234
      EE(6,1)=S2P                                                        235
      EE(7,1)=DS1P                                                       236
      EE(8,1)=DS2P                                                       237
      EE(9,1)=A1P                                                        238
      EE(10,1)=A2P                                                       239
      EE(11,1)=DA1P                                                      240
      EE(12,1)=DA2P                                                      241
      EE(1,3)= Y1Q
      EE(2,3)= Y2Q
      EE(3,3)= DY1Q
```

```
      EE(4,3)= DY2Q
      EE(5,3)= S1Q
      EE(6,3)= S2Q
      EE(7,3)= DS1Q
      EE(8,3)= DS2Q
      EE(9,3)= A1Q
      EE(10,3)= A2Q
      EE(11,3)= DA1Q
      EE(12,3)= DA2Q
      EE(1,5)=Y1C
      EE(2,5)=Y2C
      EE(3,5)=DY1C
      EE(4,5)=DY2C
      EE(5,5)=S1C
      EE(6,5)=S2C
      EE(7,5)=DS1C
      EE(8,5)=DS2C
      EE(9,5)=A1C
      EE(10,5)=A2C
      EE(11,5)=DA1C
      EE(12,5)=DA2C
      GO TO 56
C     BACKWARD 52
 52   CONTINUE
      VV(1,1)=Y1-VV(1,1)
      VV(2,1)=Y2-VV(2,1)
      VV(3,1)=DY1-VV(3,1)
      VV(4,1)=DY2-VV(4,1)
      VV(5,1)=S1-VV(5,1)
      VV(6,1)=S2-VV(6,1)
      VV(7,1)=DS1-VV(7,1)
      VV(8,1)=DS2-VV(8,1)
      VV(9,1)=A1-VV(9,1)
      VV(10,1)=A2-VV(10,1)
      VV(11,1)=DA1-VV(11,1)
      VV(12,1)=DA2-VV(12,1)
      EE(1,5)=EE(1,5)-Y1C
      EE(2,5)=EE(2,5)-Y2C
      EE(3,5)=EE(3,5)-DY1C
      EE(4,5)=EE(4,5)-DY2C
      EE(5,5)=EE(5,5)-S1C
      EE(6,5)=EE(6,5)-S2C
      EE(7,5)=EE(7,5)-DS1C
      EE(8,5)=EE(8,5)-DS2C
      EE(9,5)=EE(9,5)-A1C
      EE(10,5)=EE(10,5)-A2C
      EE(11,5)=EE(11,5)-DA1C
      EE(12,5)=EE(12,5)-DA2C
      EE(1,7)=-Y1P
      EE(2,7)=-Y2P
      EE(3,7)=-DY1P
      EE(4,7)=-DY2P
      EE(5,7)=-S1P
      EE(6,7)=-S2P
      EE(7,7)=-DS1P
      EE(8,7)=-DS2P
```

```
      EE(9,7)=-A1P                                                      305A
      EE(10,7)=-A2P                                                     305B
      EE(11,7)=-DA1P                                                    306
      EE(12,7)=-DA2P                                                    307
      EE(1,9)=-Y1Q
      EE(2,9)=-Y2Q
      EE(3,9)=-DY1Q
      EE(4,9)=-DY2Q
      EE(5,9)=-S1Q
      EE(6,9)=-S2Q
      EE(7,9)=-DS1Q
      EE(8,9)=-DS2Q
      EE(9,9)=-A1Q
      EE(10,9)=-A2Q
      EE(11,9)=-DA1Q
      EE(12,9)=-DA2Q
      EE(1,11)=-Y1T                                                     327
      EE(2,11)=-Y2T                                                     328
      EE(3,11)=-DY1T                                                    329
      EE(4,11)=-DY2T                                                    330
      EE(5,11)=-S1T                                                     331
      EE(6,11)=-S2T                                                     332
      EE(7,11)=-DS1T                                                    333
      EE(8,11)=-DS2T                                                    334
      EE(9,11)=-A1T                                                     335
      EE(10,11)=-A2T                                                    336
      EE(11,11)=-DA1T                                                   337
      EE(12,11)=-DA2T                                                   338
C     EVEN COLUMNS                                                      338
      DO 58 K=1,6                                                       339
      DO 58 L=1,6
      EE(2*L-1,2*K)=-EE(2*L,2*K-1)                                      341
      EE(2*L,2*K)=EE(2*L-1,2*K-1)                                       342
   58 CONTINUE                                                          343
      Y1=0.                                                             344
      DO 60 L=1,12                                                      345
      Y1=Y1+VV(L,1)*VV(L,1)                                             346
   60 CONTINUE                                                          347
      IF(IPRINT.GT.1)GO TO 85
      PRINT  80,(VV(L,1),L=1,12)
   80 FORMAT(/2X,6E16.8 /2X,6E16.8)
   85 CALL GAUSS(EE,12,12,VV,1,1)
      Y2=0.                                                             349
      DO 62 L=1,12                                                      350
      Y2=Y2+VV(L,1)*VV(L,1)                                             351
   62 CONTINUE                                                          352
      IF(IPRINT.GT.1)GO TO 90
      PRINT 80,(VV(L,1),L=1,12)
   90 CONTINUE
      DELP1=VV(1,1)                                                     353
      DELP2=VV(2,1)                                                     354
      DELQ1=VV(3,1)                                                     355
      DELQ2=VV(4,1)                                                     356
      DELC1=VV(5,1)                                                     357
      DELC2=VV(6,1)                                                     358
      DELR1=VV(7,1)                                                     359
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      DELR2=VV(8,1)                                                     360
      DELG1=VV(9,1)                                                      361
      DELG2=VV(10,1)                                                    362
      DELT1=VV(11,1)                                                    363
      DELT2=VV(12,1)                                                    364
      GO TO 56                                                          365
 56   RETURN                                                            366
      END                                                               367
£IBFTC GASDP   NOPRNT
      SUBROUTINE GAUSS(A,NN,N,B,MM,M)
      DIMENSION A(12,12),B(12,1),IPIVOT(12),PIVOT(12),INDEX(12,3)
       DOUBLE PRECISION A,B
      EQUIVALENCE (IROW,JROW), (ICOLUM,JCOLUM), (AMAX, T, SWAP)         F4020007
C     INITIALIZATION
   15 DO 20 J=1,N                                                       F4020012
   20 IPIVOT(J)=0                                                       F4020013
   30 DO 550 I=1,N                                                      F4020014
C     SEARCH FOR PIVOT ELEMENT                                          F4020016
   40 AMAX=0.0                                                          F4020018
   45 DO 105 J=1,N                                                      F4020019
   50 IF (IPIVOT(J)-1) 60, 105, 60                                      F4020020
   60 DO 100 K=1,N                                                      F4020021
 70   IF(IPIVOT(K)-1)80,100,740                                         F4020022
 80   IF(ABS(AMAX)-DABS(A(J,K)))85,100,100
 85   IROW=J                                                            F4020024
 90   ICOLUM=K                                                          F4020025
   95 AMAX=A(J,K)                                                       F4020026
 100  CONTINUE                                                          F4020027
  105 CONTINUE                                                          F4020028
  110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1                                   F4020029
C     INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL                 F4020031
  130  IF(IROW-ICOLUM) 140,260,140
 140  CONTINUE
  150 DO 200 L=1,N                                                      F4020035
  160 SWAP=A(IROW,L)                                                    F4020036
  170 A(IROW,L)=A(ICOLUM,L)                                             F4020037
  200 A(ICOLUM,L)=SWAP                                                  F4020038
  205 IF(M) 260, 260, 210                                               F4020039
  210 DO 250 L=1, M                                                     F4020040
  220 SWAP=B(IROW,L)                                                    F4020041
  230 B(IROW,L)=B(ICOLUM,L)                                             F4020042
  250 B(ICOLUM,L)=SWAP                                                  F4020043
  260 INDEX(I,1)=IROW                                                   F4020044
  270 INDEX(I,2)=ICOLUM                                                 F4020045
  310 PIVOT(I)=A(ICOLUM,ICOLUM)                                         F4020046
C     DIVIDE PIVOT ROW BY PIVOT ELEMENT                                 F4020049
  330 A(ICOLUM,ICOLUM)=1.0                                              F4020051
  340 DO 350 L=1,N                                                      F4020052
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT(I)                                  F4020053
  355 IF(M) 380, 380, 360                                               F4020054
  360 DO 370 L=1,M                                                      F4020055
  370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT(I)                                  F4020056
C     REDUCE NON-PIVOT ROWS                                             F4020058
  380 DO 550 L1=1,N                                                     F4020060
 390  IF(L1-ICOLUM )400, 550, 400                                       F4020061
  400 T=A(L1,ICOLUM)                                                    F4020062
```

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  420 A(L1,ICOLUM)=0.0                                                    F4020063
  430 DO 450 L=1,N                                                        F4020064
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T                                       F4020065
  455 IF(M) 550, 550, 460                                                 F4020066
  460 DO 500 L=1,M                                                        F4020067
 500  B(L1,L)=B(L1,L)-B(ICOLUM,L)*T                                       F4020068
 550  CONTINUE                                                            F4020069
C     INTERCHANGE COLUMNS                                                 F4020071
  600 DO 710 I=1,N                                                        F4020073
  610 L=N+1-I                                                             F4020074
  620 IF (INDEX(L,1)-INDEX(L,2)) 630, 710, 630                            F4020075
  630 JROW=INDEX(L,1)                                                     F4020076
  640 JCOLUM=INDEX(L,2)
  650 DO 705 K=1,N                                                        F4020078
  660 SWAP=A(K,JROW)                                                      F4020079
  670 A(K,JROW)=A(K,JCOLUM)                                               F4020080
  700 A(K,JCOLUM)=SWAP                                                    F4020081
  705 CONTINUE                                                            F4020082
  710 CONTINUE                                                            F4020083
  740 RETURN                                                              F4020084
      END
£ENTRY
    1   24   30    1   15   15    1
 .0025 .05  1.     .5
   .1000   .1000  1.0000    2
     0.0           1.0          0.1000
  1.30000000 15.00000000   4500.0
 -0.3098740794955324D 01 -0.2234099599445573D-01 -0.7810639256484553D 01
 -0.5765141780835510D 01  0.2572350419566441D 02 -0.1058417864732388D 02
 -0.4662111630903466D 04  0.1599514163750998D 05 -0.9956715300478498D 04
 -0.6787564833112370D 06  0.3570116702212894D 00  0.6071817102296290D-02
```